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Inscribed Conics and the Darboux Cubic

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Abstract. Let $\mathcal{C}$ is a inscribed conic in ABC with center P. The points of tangency of $\mathcal{C}$ with sidelines of triangle ABC are P_A, P_B, P_C. Let $A_UB_UC_U$ is the pedal triangle of the point U. $D_A = P_BP_C \cap B_UC_U, D_B = P_CP_A \cap C_UA_U, D_C = P_AP_B \cap A_UB_U$. If U lies on the Darboux cubic ($K004$), then the triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective and the perspector lies on the $\mathcal{C}$.

Keywords. Euclidean geometry, triangle geometry, barycentric coordinates, inscribed conic, pedal triangle, Darboux cubic.

1. Introduction

This paper is generalization of the note of Angel Montesdeoca [4, 10/01/2019] "La cúbica de Darboux y la elipse inscrita de Steiner".

2. Preliminaries

We shall work with homogeneous barycentric coordinates. We consider a nondegenerate triangle ABC as the reference triangle, and set up a coordinate system for points in the plane of the triangle ($a = |BC|, b = |CA|, c = |AB|$).

$$A = (1 : 0 : 0)\ , \quad B = (0 : 1 : 0)\ , \quad C = (0 : 0 : 1)$$

Definition 1. [5, §10.2] **Inscribed conic**
An **inscribed conic** is one tangent to the three sidelines of triangle ABC.

Definition 2. [5, §10.2] **Perspector of a inscribed conic**
The points of tangency of the inscribed conic with sidelines of triangle ABC form a triangle perspective with ABC at perspector, which we call **the perspector of the inscribed conic**.

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Definition 3. [1, Part 1, K004] **Darboux cubic**

The **Darboux cubic** $(K004, pK(X6, X20)$ has equation

$$K004 = \sum_{cyclic} \left((2a^2(b^2 + c^2) + (b^2 - c^2)^2 - 3a^4)x(c^2y^2 - b^2z^2)\right) = 0 \tag{1}$$

Definition 4. [6, §5.5] **Isotomic conjugate**

Two points P and $P^\bullet$ (not on any of the sidelines of the reference triangle) are said to be isotomic conjugates, if their respective traces on the sidelines are symmetric with respect to the endpoints of the corresponding sides.

Definition 5. [6, §2.2.2] **Superior and inferior**

The homotheties $\mathsf{h}(G, -2)$ and $\mathsf{h}(G, -1/2)$ are called the superior and inferior operations respectively. Thus, $P^{\mathbf{S}}$ and $P^{\mathbf{I}}$ are the points dividing P and the centroid G according to the ratios

$$PG : GP^{\mathbf{S}} = 1 : 2, \quad PG : GP^{\mathbf{I}} = 2 : 1$$

The triangle $M_aM_bM_c$ is the image of ABC under the inferior operation; it is called the **inferior triangle** (the points M_a, M_B, M_C are midpoints of the sides BC, CA, AB respectively).

Definition 6. [5, §2.2.2] **Pedal triangle**

The **pedals** of a point P are the intersections of the sidelines with the corresponding perpendiculars through P . They form the **pedal triangle** of P .

3. Inscribed conic with center P

Let P has homogeneous barycentric coordinates $(p : q : r)$. Let $\mathcal{C}$ be inscribed conic with center P and perspector Q.

The isotomic conjugate $Q^\bullet$ of the perspector Q is superior of the center P (see Figure 1).

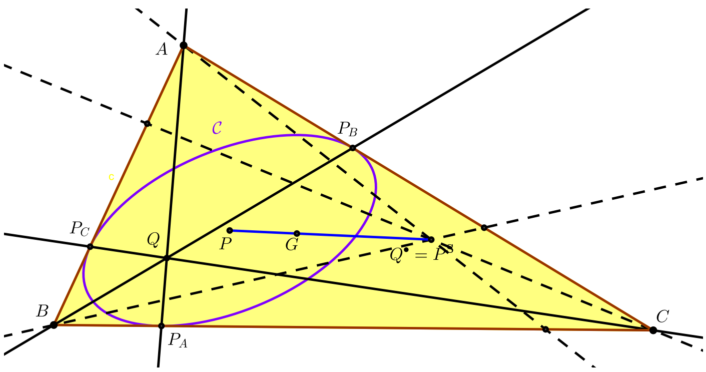

Figure 1. Inscribed conic with center P

The point $P^{\mathbf{S}}$ is the image of P under the superior operation $\mathsf{h}(G, -2)$.

$$P^{\mathbf{S}} = 3G - 2P = (-p + q + r : p - q + r : p + q - r) = Q^\bullet$$

Q is the isotomic conjugate of the point $Q^\bullet = P^{\mathbf{S}}$.

$$\begin{aligned} Q &= \left(\frac{1}{-p + q + r} : \frac{1}{p - q + r} : \frac{1}{p + q - r}\right) \\ &= ((p + q - r)(p - q + r) : (p + q - r)(-p + q + r) : (p - q + r)(-p + q + r)) \end{aligned}$$

The traces of Q are the points of tangency of the inscribed conic with center P and sidelines of triangle ABC

$$(2)\qquad \begin{aligned} P_A &= (0 : p+q-r : p-q+r) \\ P_B &= (-p-q+r : 0 : p-q-r) \\ P_C &= (-p+q-r : p-q-r : 0) \end{aligned}$$

The equation of the inscribed conic $\mathcal{C}$ with center P and perspector Q is

$$(3)\qquad \sum_{cyclic} \big((p^2-2pq+q^2-2pr+2qr+r^2)x^2 + 2(-p^2+q^2-2qr+r^2)yz\big) = 0$$

4. Perspective triangles

4.1. U lies on Darboux cubic. .

Theorem 1. *Let $\mathcal{C}$ is a inscribed conic in ABC with center P. The points of tangency of $\mathcal{C}$ with sidelines of triangle ABC are P_A, P_B, P_C. Let $A_UB_UC_U$ is the pedal triangle of the point U. $D_A = P_BP_C \cap B_UC_U, D_B = P_CP_A \cap C_UA_U, D_C = P_AP_B \cap A_UB_U$. If U lies on the Darboux cubic (K004), then:*

(a) the triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective

(b) the perspector Z lies on the $\mathcal{C}$

(c) A, D_B, D_C are collinear, also B, D_C, D_A and C, D_A, D_B

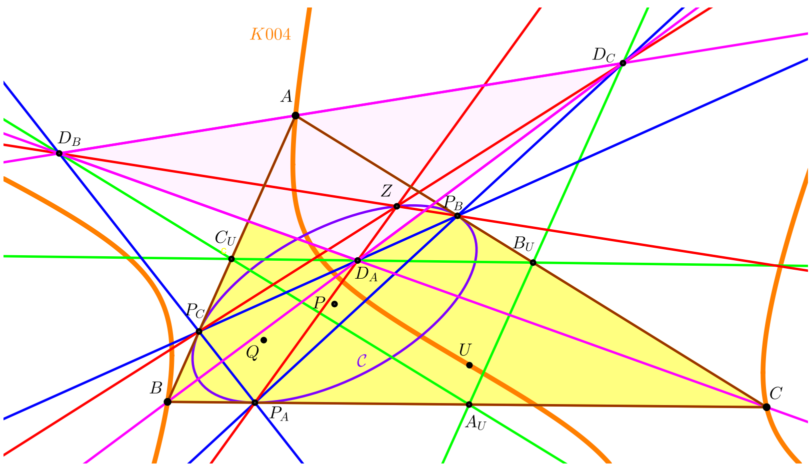

FIGURE 2. Perspective triangles

Proof. Let the point $U = (u : v : w)$. The pedals of U are the points:

$$(4)\qquad \begin{aligned} A_U &= (0 : (b^2-c^2)u + a^2(u+2v) : (-b^2+c^2)u + a^2(u+2w)) \\ B_U &= ((a^2-c^2)v + b^2(2u+v) : 0 : (-a^2+c^2)v + b^2(v+2w)) \\ C_U &= ((a^2-b^2)w + c^2(2u+w) : (-a^2+b^2)w + c^2(2v+w) : 0) \end{aligned}$$

The equation of the line joined the points A_U and B_U is (see [2])

(5)

$$
\begin{aligned}
A_UB_U : &\begin{vmatrix} 0 & (b^2-c^2)u+a^2(u+2v) & (-b^2+c^2)u+a^2(u+2w) \\ (a^2-c^2)v+b^2(2u+v) & 0 & (-a^2+c^2)v+b^2(v+2w) \\ x & y & z \end{vmatrix} \\
&= ((b^2-c^2)u+a^2(u+2v))((c^2-a^2)v+b^2(v+2w))x \\
&+ ((a^2-c^2)v+b^2(2u+v))((-b^2+c^2)u+a^2(u+2w))y \\
&+ ((a^2-c^2)v+b^2(2u+v))((-b^2+c^2)u-a^2(u+2v))z = 0
\end{aligned}
$$

The equation of the line joined the points P_A and P_B (2) is

$$
\begin{aligned}
P_AP_B : &\begin{vmatrix} 0 & p+q-r & p-q+r \\ -p-q+r & 0 & p-q-r \\ x & y & z \end{vmatrix} \\
&= (p-q-r)x+(-p+q-r)y+(p+q-r)z = 0
\end{aligned} \tag{6}
$$

The point $D_C = P_AP_B \cap A_UB_U$ has coordinates

$$
\begin{aligned}
D_C = &(-(a^6vw((p-q)(v-w)+r(2u+3v+w)))+(b^2-c^2)u(c^4v(r(u-v)-p(u+3v+2w) \\
&+q(u+3v+2w))-b^4w(q(u-w)+p(-u+w)+r(u+2v+3w))-b^2c^2(q(uv-v^2-uw-6vw-3w^2) \\
&+r(uv+3v^2-uw+6vw+w^2)+p(v^2+6vw+3w^2+u(-v+w))))+a^4(c^2v(p(u^2+3uv+4v^2 \\
&+2uw+2vw-2w^2)-q(u^2+3uv+4v^2+2uw+2vw-2w^2)+r(-u^2+uv+4v^2+4uw+6vw+2w^2)) \\
&+b^2w(p(u^2+3uw+2v(v+w))-q(u^2+3uw+2v(v+w))+r(-u^2+6v(v+w)+u(2v+w)))) \\
&-a^2(c^4v(p(2u^2+6uv+4v^2+4uw+vw-w^2)-q(2u^2+6uv+4v^2+4uw+vw-w^2) \\
&+r(-2u^2+4v^2+3vw+w^2+2u(v+w)))+b^4w(p(2u^2+2uw+v(v+3w))-q(2u^2+2uw+v(v+3w)) \\
&-r(2u^2+2u(v+w)-v(3v+5w)))+2b^2c^2(-(r(u^2(v+w)+u(v^2-w^2)-v(2v^2+5vw+w^2))) \\
&+p(u^2(v+w)+u(v^2+6vw+3w^2)+v(2v^2+7vw+3w^2))-q(u^2(v+w)+u(v^2+6vw+3w^2) \\
&+v(2v^2+7vw+3w^2)))) \\
&:(-p+q-r)(a^6v(v-w)w-(b^2-c^2)u(c^4(u-v)v+b^4(u-w)w+b^2c^2(v^2+w^2+3u(v+w))) \\
&+a^4(-(b^2w(u^2+3uw+2v(v+w)))+c^2v(u^2+3uv+2w(v+w)))+a^2(2b^2c^2(u-v)(u-w)(v-w) \\
&-c^4v(2u^2+2uv+w(3v+w))+b^4w(2u^2+2uw+v(v+3w)))) \\
&:a^6vw(p(-v+w)+q(2u+3v+w)-r(2u+3v+w))+(b^2-c^2)u(b^4w((-q+r)(u-w)+p(3u+2v+w)) \\
&-c^4v((q-r)(u-v)+p(u+3v+2w))+b^2c^2((-q+r)(v^2+w^2+3u(v+w))+p(3v^2-w^2+u(5v+w)))) \\
&+a^4(b^2w(-(p(u^2+u(6v-w)+2v(v+w)))+(q-r)(3u^2-2v(v+w)+u(4v+w))) \\
&+c^2v(p(u^2+3uv+4v^2+2uw+2vw-2w^2)+(q-r)(u^2-u(v+4w)-2(2v^2+3vw+w^2)))) \\
&+a^2(-(c^4v(p(2u^2+6uv+4v^2+4uw+vw-w^2)+(q-r)(2u^2-4v^2-3vw-w^2-2u(v+w)))) \\
&+b^4w((-q+r)(2u^2+v(v-w)+2u(3v+w))+p(-2u^2+u(4v-2w)+v(3v+w)))+2b^2c^2(p(u^2(v+w) \\
&+u(5v^2-w^2)+v(2v^2-vw-w^2))-(q-r)(3u^2(v+w)+v(2v^2+vw+w^2)+u(7v^2+6vw+w^2)))))
\end{aligned}
$$

The points D_A, D_B receive similarly.

The triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective if and only if the lines D_AP_A, D_BP_B, D_CP_C are concurrent. The lines $D_AP_A : a_xx + a_yy + a_zz = 0, D_BP_B : b_xx + b_yy + b_zz = 0, D_CP_C : c_xx + c_yy + c_zz = 0$ are concurrent if and only if (see [2], [5, §4.3]):

$$\mathit{\Delta}=\begin{vmatrix}D_AP_A\\D_BP_B\\D_CP_C\end{vmatrix}=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}=0$$

(7)

$$\begin{aligned}\mathit{\Delta}=8\prod_{ciclic}(-p+q+r)\prod_{ciclic}(((-b^2+c^2)p+a^2(q-r))u-a^2(p-q+r)v+a^2(p+q-r)w)\\ \cdot\sum_{cyclic}\big((2a^2(b^2+c^2)+(b^2-c^2)^2-3a^4)u(c^2v^2-b^2w^2)\big)\end{aligned}$$

The last factor manifest that $\mathit{\Delta}=0$ if U lies on $K004$ (1).

The point $Z=(z_x:z_y:z_z)=D_AD_B\cap D_AD_C$. Substituting coordinates of Z in equation of $\mathcal{C}$ (3), receive

(8)

$$\begin{aligned}&\sum_{cyclic}\big((p^2-2pq+q^2-2pr+2qr+r^2)z_x^2+2(-p^2+q^2-2qr+r^2)z_yz_z\big)\\ &=-\frac{1}{2}((p+q-r)(p-q+r)((b^2-c^2)u+a^2(u+2v))((-b^2+c^2)u+a^2(u+2w)))\mathit{\Delta}\\ &=0\quad\text{if } U \text{ lies on } K004 \text{ and } \mathit{\Delta}=0\end{aligned}$$

The point A lies on the line $D_BD_C:\ d_xx+d_yy+d_zz=0$ if and only if $d_x=0$.

$$d_x=(-p+q+r)\sum_{cyclic}\big((2a^2(b^2+c^2)+(b^2-c^2)^2-3a^4)u(c^2v^2-b^2w^2)\big)$$

and $d_x=0,$ if U lies on $K004$ (1). □

In the next table we consider some interesting examples for the points $Z=(z_x:z_y:z_z)$ [2] and $U\in K004$ [3]

[2] To obtain z_y, in z_x substitute $a;b;c;p;q;r$ for $b;c;a;q;r;p$ respectively, and to obtain z_z, in z_y substitute $a;b;c;p;q;r$ for $b;c;a;q;r;p$ respectively.

[3] The Darboux cubic $K004$ contains (see [1]) the points X_i for i = 1, 3, 4, 20, 40, 64, 84, 1490, 1498, 2130, 2131, 3182, 3183, 3345, 3346, 3347, 3348, 3353, 3354, 3355, 3472, 3473, 3637, see Encyclopedia of Triangle Centers [3]

Point U	The first coordinate z_x of $Z = (z_x : z_y : z_z)$
$I = X_1$	$(p-q-r)(bp-cp+a(-q+r))^2$
$O = X_3$	$(-p+q+r)(q-r)^2$
$H = X_4$	$(p-q-r)(b^2p-c^2p+a^2(-q+r))^2$
X_{20}	$(-p+q+r)(-a^2+b^2+c^2)^2(b^2p-c^2p+a^2(q-r))^2$
X_{40}	$(-p+q+r)(-a+b+c)^2(bp-cp+a(q-r))^2$
X_{64}	$(-p+q+r)(2a^2(b^2-c^2)p+a^4(q-r)-(b^2-c^2)(b^2(2p+q-r)+c^2(2p-q+r)))^2$
X_{84}	$(-p+q+r)(a^2(b-c)p-(b-c)(b+c)^2p+a^3(q-r)-a(b-c)^2(q-r))^2$
X_{1490}	$(p-q-r)(a^3+a^2(b+c)-(b-c)^2(b+c)-a(b+c)^2)^2$
	$.(a^2(-b+c)p+(b-c)(b+c)^2p+a^3(q-r)-a(b-c)^2(q-r))^2$
X_{1498}	$(p-q-r)(-3a^4+(b^2-c^2)^2+2a^2(b^2+c^2))^2$
	$.(2a^2(-b^2+c^2)p+a^4(q-r)+(b^2-c^2)(c^2(2p+q-r)+b^2(2p-q+r)))^2$
X_{3182}	$(-p+q+r)(a^6-2a^5(b+c)-a^4(b+c)^2+(b-c)^2(b+c)^4-a^2(b^2-c^2)^2+4a^3(b^3+c^3)$
	$-2a(b^5-b^4c-bc^4+c^5))^2(a^6(b-c)p-3a^4(b-c)(b+c)^2p-(b-c)^5(b+c)^2p+a^2(b+c)^2$
	$.(3b^3-5b^2c+5bc^2-3c^3)p+a^7(q-r)-3a^5(b-c)^2(q-r)-a(b-c)^2(b+c)^4(q-r)$
	$+a^3(b-c)^2(3b^2+2bc+3c^2)(q-r))^2$

4.2. $(-p+q+r)=0$. .

The first factor from determinant Δ (7) is

$$\alpha = \prod_{ciclic}(-p+q+r) = (-p+q+r)(p-q+r)(p+q-r)$$

Let for example $(-p+q+r)=0$. Then $\alpha = 0$ and $\Delta = 0$. The points $P=(p,q,r)$ (in absolute barycentric coordinates, $p+q+r=1$) are $\left(\frac{1}{2} : \frac{1}{2} - t : t\right)$, for $t \in R$. The locus of points P is the line, that joint the midpoints M_c, M_b of the segments AB, AC respectively.

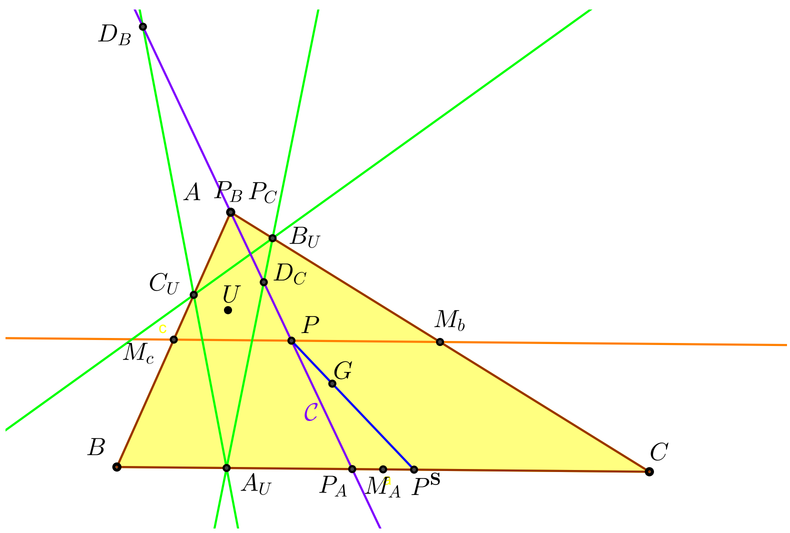

FIGURE 3. $P \in M_bM_c$

Theorem 2. *Let $\mathcal{C}$ is a inscribed conic in ABC with center P. The points of tangency of $\mathcal{C}$ with sidelines of triangle ABC are P_A, P_B, P_C. Let $A_UB_UC_U$ is the pedal triangle of the point U. $D_A = P_BP_C \cap B_UC_U, D_B = P_CP_A \cap C_UA_U, D_C = P_AP_B \cap A_UB_U$. If P lies on the sidelines of the inferior triangle $M_aM_bM_c$, then:*

(a) the inscribed conic $\mathcal{C}$ degenerate to the line AP (or BP, or CP).

(b) $D_AD_BD_C$ and $P_AP_BP_C$ are perspective, and the perspector is P_A (or P_B, or P_C).

Proof. Let $P \in M_bM_c$. Then $P^{\mathbf{S}}$ lies on the line BC and $Q = A$. The conic $\mathcal{C}$ degenerate to a line AP, $P_A = AP \cap BC, P_B = P_C = A$. The points $D_B \in AP, D_C \in AP$, follow $D_BP_B = AP, D_CP_C = AP$ and the lines D_AP_A, D_BP_B, D_CP_C intersect in P_A. Similarly: if $P \in M_aM_b$, the lines D_AP_A, D_BP_B, D_CP_C intersect in P_C, if $P \in M_cM_a$, the lines D_AP_A, D_BP_B, D_CP_C intersect in P_B □

4.3. $(-b^2p + c^2p + a^2q - a^2r)u + (-a^2p + a^2q - a^2r)v + (a^2p + a^2q - a^2r)w = 0$. .
The second factor from determinant Δ (7) is

$$\beta = \prod_{ciclic} (-b^2p + c^2p + a^2q - a^2r)u + (-a^2p + a^2q - a^2r)v + (a^2p + a^2q - a^2r)w$$

Let for example $\beta_1 = (-b^2p + c^2p + a^2q - a^2r)u + (-a^2p + a^2q - a^2r)v + (a^2p + a^2q - a^2r)w = 0$. Then $\beta = 0$ and $\Delta = 0$. Let the line $\mathcal{L}_A$ has equation

$$\mathcal{L}_A : (-b^2p + c^2p + a^2q - a^2r)x + (-a^2p + a^2q - a^2r)y + (a^2p + a^2q - a^2r)z = 0.$$

This line is perpendicular to BC at P_A (2).

$\beta_1 = 0$ if and only if the point $U = (u : v : w)$ lies on the line $\mathcal{L}_A$.

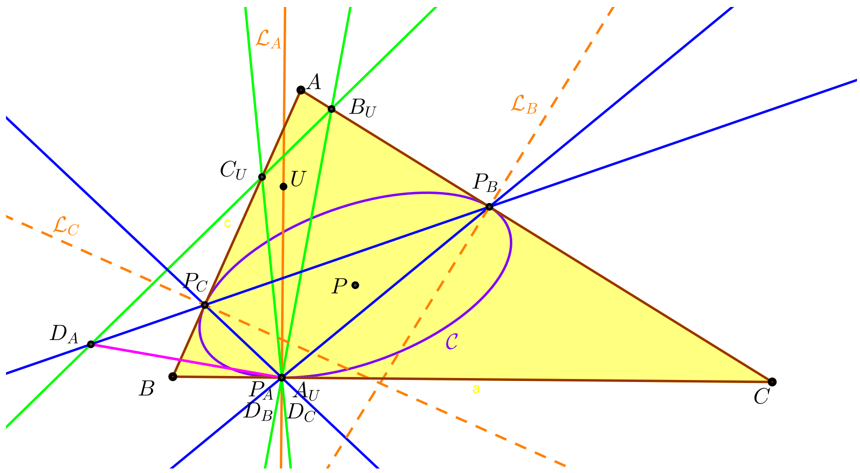

FIGURE 4. $U \in \mathcal{L}_A$

Theorem 3. *Let $\mathcal{C}$ is a inscribed conic in ABC with center P. The points of tangency of $\mathcal{C}$ with sidelines of triangle ABC are P_A, P_B, P_C. Let $A_UB_UC_U$ is the pedal triangle of the point U. $D_A = P_BP_C \cap B_UC_U, D_B = P_CP_A \cap C_UA_U, D_C = P_AP_B \cap A_UB_U$. If U lies on the line $\mathcal{L}_A$, perpendicular to BC at P_A, (or on the line $\mathcal{L}_B$, perpendicular to CA at P_B, or on the line $\mathcal{L}_C$, perpendicular to AB at P_C), then:*

(a) the triangle $D_AD_BD_C$ is degenerate

(b) the triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective at P_A (or P_B, or P_C).

Proof. Let $U \in \mathcal{L}_A$. $A_U = P_A$. The points $D_B = D_C = P_A$, follow the lines D_AP_A, D_BP_B, D_CP_C intersect at P_A. Similarly for cases $U \in \mathcal{L}_B$ and $U \in \mathcal{L}_C$. □

Conclusion 1. *Let $\mathcal{C}$ is a inscribed conic in ABC with center P. The points of tangency of $\mathcal{C}$ with sidelines of triangle ABC are P_A, P_B, P_C. Let $A_UB_UC_U$ is the pedal triangle of the point U. $D_A = P_BP_C \cap B_UC_U, D_B = P_CP_A \cap C_UA_U, D_C = P_AP_B \cap A_UB_U$.*

(a) If P lies on the sidelines of the inferior triangle $M_aM_bM_c$, then the triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective at P_A (or P_B, or P_C).

(b) If U lies on the line $\mathcal{L}_A$, perpendicular to BC at P_A, (or on the line $\mathcal{L}_B$, perpendicular to CA at P_B, or on the line $\mathcal{L}_C$, perpendicular to AB at P_C), then the triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective at P_A (or P_B, or P_C).

(c) If P not on the sidelines of the inferior triangle $M_aM_bM_c$, and U not on the lines $\mathcal{L}_A, \mathcal{L}_B, \mathcal{L}_C$, the triangles $D_AD_BD_C$ and $P_AP_BP_C$ are perspective, and the perspector Z lies on the $\mathcal{C}$, if and only if U lies on the Darboux cubic ($K004$).

5. EXAMPLES

5.1. $P = I, \mathcal{C}$ is the incircle.

Point U	$Z = X_i$, or the first coordinate $[z_x]$ of $Z = (z_x : z_y : z_z)$
$O = X_3$	X_{11}
$H = X_4$	X_{3022}
X_{20}	X_{1364}
X_{40}	X_{3022}
X_{64}	X_{3318}
X_{84}	X_{1364}
X_{1490}	X_{3318}
X_{1498}	$[(a-b-c)^3(b-c)^2(-3a^4+(b^2-c^2)^2+2a^2(b^2+c^2))^2]$
X_{3182}	$[-a^2(a-b-c)(b-c)^2(-a^2+b^2+c^2)^2$
	$.(a^6-2a^5(b+c)-a^4(b+c)^2+(b-c)^2(b+c)^4-a^2(b^2-c^2)^2+4a^3(b^3+c^3)-2a(b^5-b^4c-bc^4+c^5))^2]$
X_{3345}	$[(a-b-c)^3(b-c)^2(-3a^4+(b^2-c^2)^2+2a^2(b^2+c^2))^2]$
X_{3346}	$[-a^2(a-b-c)(b-c)^2(-a^2+b^2+c^2)^2$
	$.(a^6-2a^5(b+c)-a^4(b+c)^2+(b-c)^2(b+c)^4-a^2(b^2-c^2)^2+4a^3(b^3+c^3)-2a(b^5-b^4c-bc^4+c^5))^2]$
X_{3347}	$[-a^2(a-b-c)(b-c)^2(a^3+a^2(b+c)-(b-c)^2(b+c)-a(b+c)^2)^2$
	$.(a^8-4a^6(b^2+c^2)-4a^2(b^2-c^2)^2(b^2+c^2)+(b^2-c^2)^2(b^4+6b^2c^2+c^4)+a^4(6b^4-4b^2c^2+6c^4))^2]$

5.2. $P = G, \mathcal{C}$ is the Steiner in-ellipse.

See note of Angel Montesdeoca [4, 10/01/2019] "La cúbica de Darboux y la elipse inscrita de Steiner".

5.3. $P = X_5$, Nine-point center, $\mathcal{C}$ is the in-ellipse with foci O and H.

Point U	The first coordinate $[z_x]$ of $Z = (z_x : z_y : z_z)$
$I = X_1$	$[-a^2(b-c)^2(a^2-b^2-c^2)(2abc-a^2(b+c)+(b-c)^2(b+c))^2]$
$O = X_3$	$[a^2(b-c)^2(b+c)^2(a^2-b^2-c^2)^3]$
$H = X_4$	$[a^2(b-c)^2(b+c)^2(a^2+b^2-c^2)(a^2-b^2+c^2)]$
X_{20}	$[a^2(b-c)^2(b+c)^2(a^2-b^2-c^2)^3]$
X_{40}	$[a^2(b-c)^2(-a+b+c)^4(a^2-b^2-c^2)(2abc+a^2(b+c)-(b-c)^2(b+c))^2]$
X_{64}	$[a^2(b-c)^2(b+c)^2(a^2-b^2-c^2)^3]$
X_{84}	$[a^2(b-c)^2(a^2-b^2-c^2)(a^5(b+c)-(b-c)^2(b+c)^4-a^4(b^2+c^2)-2a^3(b^3+c^3)$
	$+2a^2(b^4+b^3c+bc^3+c^4)+a(b^5-b^4c-bc^4+c^5))^2]$
X_{1490}	$[-a^2(b-c)^2(a^2-b^2-c^2)(a^3+a^2(b+c)-(b-c)^2(b+c)-a(b+c)^2)^2(a^5(b+c)$
	$+(b-c)^2(b+c)^4+a^4(b^2+c^2)-2a^3(b^3+c^3)-2a^2(b^4+b^3c+bc^3+c^4)+a(b^5-b^4c-bc^4+c^5))^2]$
X_{1498}	$[-a^2(b-c)^2(b+c)^2(a^2-b^2-c^2)^3(a^4-3(b^2-c^2)^2+2a^2(b^2+c^2))^2(-3a^4+(b^2-c^2)^2+2a^2(b^2+c^2))^2]$
X_{3346}	$[a^2(b-c)^2(b+c)^2(a^2-b^2-c^2)^3(a^{1}2+(b^2-c^2)^6+2a^{1}0(b^2+c^2)$
	$+a^8(-17b^4+2b^2c^2-17c^4)-a^4(b^2-c^2)^2(17b^4+30b^2c^2+17c^4)+2a^2(b^2-c^2)^2$
	$.(b^6+7b^4c^2+7b^2c^4+c^6)+4a^6(7b^6-3b^4c^2-3b^2c^4+7c^6))^2]$

5.4. $P = X_{597}$, midpoint of X_2 and X_6, $\mathcal{C}$ is the Lemoine in-ellipse with foci G and K.

Point U	The first coordinate $[z_x]$ of $Z = (z_x : z_y : z_z)$
$I = X_1$	$[(b-c)^2(4a^2+b^2+c^2-3a(b+c))^2(a^2-2(b^2+c^2))]$
$O = X_3$	$[(b-c)^2(b+c)^2(-a^2+2(b^2+c^2))]$
$H = X_4$	$[(b-c)^2(b+c)^2(-a^2+2(b^2+c^2))]$
X_{20}	$[(b-c)^2(b+c)^2(-a^2+b^2+c^2)^2(7a^2+b^2+c^2)^2(-a^2+2(b^2+c^2))]$
X_{40}	$[-(b-c)^2(-a+b+c)^2(4a^2+b^2+c^2+3a(b+c))^2(a^2-2(b^2+c^2))]$
X_{64}	$[(b-c)^2(b+c)^2(-a^2+2(b^2+c^2))(-11a^4+5b^4-2b^2c^2+5c^4+6a^2(b^2+c^2))^2]$
X_{84}	$[(b-c)^2(-a^2+2(b^2+c^2))(-4a^4-3a^3(b+c)+3a(b-c)^2(b+c)$ $+(b+c)^2(b^2+c^2)+a^2(3b^2+8bc+3c^2))^2]$
X_{1490}	$[-(b-c)^2(a^3+a^2(b+c)-(b-c)^2(b+c)-a(b+c)^2)^2(-a^2+2(b^2+c^2))$ $.(-4a^4+3a^3(b+c)-3a(b-c)^2(b+c)+(b+c)^2(b^2+c^2)+a^2(3b^2+8bc+3c^2))^2]$
X_{1498}	$[-(b-c)^2(b+c)^2(-a^2+2(b^2+c^2))(5a^4+b^4-10b^2c^2+c^4-6a^2(b^2+c^2))^2$ $.(-3a^4+(b^2-c^2)^2+2a^2(b^2+c^2))^2]$

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