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Using the affine and projective methods to prove and extend Dao's theorem

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Abstract. We refer to the affine and projective methods in order to prove and extend the Dao's generalization of Gauss-Newton theorem.

Keywords. Gauss-Newton theorem, Dao's theorem, the affine method, the projective method, proof.

1. Introduction

The Gauss-Newton's theorem is a nice and famous theorem of Euclidean geometry. This theorem is stated as follows :

Theorem 1.1. (*Gauss-Newton [1]*). *Given a triangle ABC. Line d meets three sidelines BC, CA, AB of triangle ABC at A_1, B_1, C_1, respectively. Let A_2, B_2, C_2 be midpoints of AA_1, BB_1, CC_1 then A_2, B_2, C_2 are collinear.*

Some proofs of the Gauss-Newton theorem are in [1]
O. T. Dao expanded the Gauss-Newton theorem as follows:

Theorem 1.2. (*O. T. Dao*). *Given a triangle ABC. Line d meets three sidelines BC, CA, AB of the triangle ABC at A_1, B_1, C_1, respectively. Let P be a point on the plane, EFG be a cevian triangle of the point P. Lines AA_1, BB_1, CC_1 meet three sidelines of triangle EFG at A_2, B_2, C_2 then A_2, B_2, C_2 are collinear.*

When P is the centroid of ABC, this theorem is the Gauss-Newton theorem.
A synthetic proof is given by Tel Cohv. See [2].

2. Using the affine and projective methods to prove Theorem 2

Solution 1 *(The projective method)*

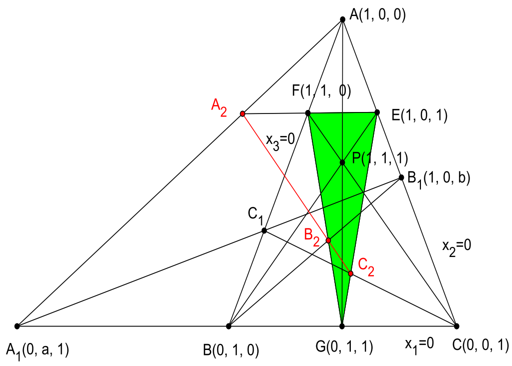

FIGURE 1. The projective method

Consider the projective target $\{A,\ B,\ C\ ;\ P\}$. We have

$$A = (1,\ 0,\ 0),\ B = (0,\ 1,\ 0),\ C = (0,\ 0,\ 1);\ P = (1,\ 1,\ 1).$$

The coordinates of the equation of the line AB are of the form

$$\left[\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix},\ \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix},\ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right] = [0,\ 0,\ 1]$$

Thus, the equation of the line AB is of the form : $x_3 = 0$.
Similarly, the equation of the line BC is of the form $x_1 = 0$.
The equation of the line CA is of the form $x_2 = 0$.
Since A_1 is on the line BC, the coordinates of the point A_1 are of the form $A_1 = (0,\ a,\ 1)$.
Since B_1 is on the line CA, the coordinates of the point B_1 are of the form $B_1 = (1,\ 0,\ b)$.
The coordinates of the equation of the line A_1B_1 are of the form

$$\left[\begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix},\ \begin{vmatrix} 1 & 0 \\ b & 1 \end{vmatrix},\ \begin{vmatrix} 0 & a \\ 1 & 0 \end{vmatrix}\right] = [ab,\ 1,\ -a]$$

Since $C_1 = A_1B_1 \cap AB$, the coordinates of the point C_1 satisfy the system of equations

$$\begin{cases} abx_1 + x_2 - ax_3 = 0 \\ x_3 = 0 \end{cases}$$

Thus, $C_1 = (1,\ -ab,\ 0)$.
The coordinates of the equation of the line CC_1 are of the form

$$\left[\begin{vmatrix} 0 & 1 \\ -ab & 0 \end{vmatrix},\ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix},\ \begin{vmatrix} 0 & 0 \\ 1 & -ab \end{vmatrix}\right] = [ab,\ 1,\ 0]$$

Thus, the equation of the line CC_1 is of the form : $abx_1 + x_2 = 0$.
The coordinates of the equation of the line GE are of the form

$$\left[\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix},\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right] = [1,\ 1,\ -1]$$

Thus, the equation of the line GE is of the form $x_1 + x_2 - x_3 = 0$.
Since $C_2 = GE \cap CC_1$, the coordinates of the point C_2 satisfy the system of equations :

$$\begin{cases} abx_1 + x_2 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$$

Thus, $C_2 = (1, -ab, 1 - ab)$.
Similarly, the equation of the line GF is of the form : $x_1 - x_2 + x_3 = 0$.
The coordinates of the line BB_1 are of the form

$$\left[\begin{vmatrix} 1 & 0 \\ 0 & b \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ b & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right] = [b, 0, -1]$$

Thus, the equation of the line BB_1 is of the form : $bx_1 - x_3 = 0$.
Since $B_2 = GF \cap BB_1$, the coordinates of the point B_2 satisfy the system of equations :

$$\begin{cases} bx_1 - x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases}$$

Thus, $B_2 = (1, 1+b, b)$.
Similarly, the equation of the line EF is of the form :

$$-x_1 + x_2 + x_3 = 0.$$

The coordinates of the equation of the line AA_1 are of the form

$$\left[\begin{vmatrix} 0 & 0 \\ a & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & a \end{vmatrix} \right] = [0, -1, a]$$

Thus, the equation of the line AA_1 is of the form :

$$-x_2 + ax_3 = 0.$$

Since $A_2 = EF \cap AA_1$, the coordinates of the point A_2 satisfy the system of equations :

$$\begin{cases} -x_2 + ax_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{cases}$$

Thus, $A_2 = (a + 1, a, 1)$.

Consider the determinant $\Delta = \begin{vmatrix} 1 & 1+b & b \\ 1 & -ab & 1-ab \\ a+1 & a & 1 \end{vmatrix}$, we have:

$$\begin{aligned} \Delta &= -ab.1 - a(1-ab) - 1.((1+b).1 - ab) + (a+1)((1+b) - ab) \\ &= -ab - a + a^2b - 1 - b + ab + (a + ab - a^2b + 1 + b - ab) \\ &= 0. \end{aligned}$$

Thus, A_2, B_2, C_2 are collinear.
Solution 2 *(the affine method)*

Considering the affine coordinate system $\{B ; BC, BA\}$, we have :

$$B = (0, 0),\ C = (1, 0),\ A = (0, 1).$$

The equation of the line AC is of the form :

$$\tfrac{x-1}{1-0} = \tfrac{y-0}{0-1} \Leftrightarrow -1(x - 1) = y \Leftrightarrow x + y - 1 = 0.$$

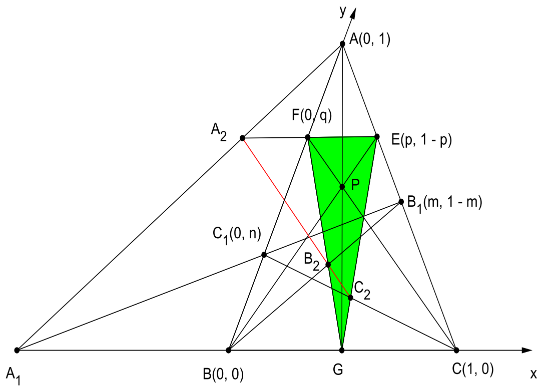

Figure 2. The affine method

The equation of the line B_1C_1 is of the form

$$\frac{x-0}{0-m} = \frac{y-n}{n+m-1} \Leftrightarrow (m+n-1)x + my - mn = 0.$$

Since $A_1 = B_1C_1 \cap BC$, the coordinates of the point A_1 satisfy the system of equations

$$\begin{cases} (m+n-1)x + my - mn = 0 \\ y = 0 \end{cases}$$

Thus, $A_1\left(\frac{mn}{m+n-1},\ 0\right)$.

The equation of the line AA_1 is of the form

$$\frac{x - \frac{mn}{m+n-1}}{\frac{mn}{m+n-1} - 0} = \frac{y-0}{0-1} \Leftrightarrow -1\left(x - \frac{mn}{m+n-1}\right) = \frac{mn}{m+n-1}y$$
$$\Leftrightarrow (m+n-1)x + mny - mn = 0.$$

The equation of the line BB_1 is of the form

$$(1-m)x - my = 0.$$

The equation of the line BE is of the form

$$(1-p)x - py = 0.$$

The equation of the line CC_1 is of the form

$$\frac{x-1}{1-0} = \frac{y-0}{0-n} \Leftrightarrow nx + y - n = 0.$$

The equation of the line CF is of the form

$$\frac{x-1}{1-0} = \frac{y-0}{0-q} \Leftrightarrow qx + y - q = 0.$$

The equation of the line EF is of the form

$$\frac{x-p}{p-0} = \frac{y-(1-p)}{(1-p)-q} \Leftrightarrow (1-p-q)(x-p) - p(y-(1-p)) = 0$$
$$\Leftrightarrow (1-p-q)x - py + pq = 0.$$

Since $A_2 = EF \cap AA_1$, the coordinates of the point A_2 satisfy the system of equations:

$$\begin{cases} (m+n-1)x + mny - mn = 0 \\ (1-p-q)x - py + pq = 0 \end{cases}$$

Solving this system, we have

$$A_2 = \left(\frac{mnp(q-1)}{mnp+mnq-mn-mp-np+p}, \frac{mnp+mnq-mpq-npq-mn+pq}{mnp+mnq-mn-mp-np+p}\right).$$

Since $P = BE \cap CF$, the coordinates of the point P satisfy the system of equations:

$$\begin{cases} (1-p)x - py = 0 \\ qx + y - q = 0 \end{cases}$$

Solving this system, we have :

$$P = \left(\frac{qp}{pq-p+1}, \frac{q(1-p)}{pq-p+1}\right).$$

The equation of the line AP is of the form :

$$\frac{x - \frac{qp}{pq-p+1}}{\frac{qp}{pq-p+1}} = \frac{y - \frac{q(1-p)}{pq-p+1}}{\frac{q(1-p)}{pq-p+1} - 1}.$$

Simplifying the equation of the line AP, we have :

$$(2pq - p - q + 1)x + pqy - pq = 0.$$

Since $G = AP \cap BC$, the coordinates of the point G satisfy the system of equation :

$$\begin{cases} (2pq - p - q + 1)x + pqy - pq = 0 \\ y = 0 \end{cases}$$

Thus, $G = \left(\frac{pq}{2pq-p-q+1}, 0\right)$.
The equation of the line GE is of the form

$$\frac{x - \frac{pq}{2pq-p-q+1}}{\frac{pq}{2pq-p-q+1} - p} = \frac{y-0}{0-1+p} \Leftrightarrow (2pq - p - q + 1)x - (2pq-p)y - pq = 0$$

The equation of the line GF is of the form

$$\frac{x - \frac{pq}{2pq-p-q+1}}{\frac{pq}{2pq-p-q+1} - 0} = \frac{y-0}{0-q} \Leftrightarrow (2pq - p - q + 1)x + py - pq = 0.$$

Since $C_2 = GE \cap B_1C_1$, the coordinates of the point C_2 satisfy the system of equations

$$\begin{cases} (m + n - 1)x + my - mn = 0 \\ (2pq - p - q + 1)x - (2pq-p)y - pq = 0 \end{cases}$$

Thus

$$C_2 = \left(\frac{mp(2nq - n - q)}{2npq + mq - np - 2pq - m + p}, -\frac{2mnpq - mnp - mnq - mpq - npq + mn + pq}{2npq + mq - np - 2pq - m + p}\right).$$

Since $B_2 = GF \cap B_1C_1$, the coordinates of the point B_2 satisfy the system of equations

$$\begin{cases} (2pq - p - q + 1)x + py - pq = 0 \\ (m + n - 1)x + my - mn = 0 \end{cases}$$

Thus, $B_2 = \left(\frac{mp(2nq-n-q)}{2npq+mq-np-2pq-m+p}, -\frac{2mnpq-mnp-mnq-mpq-npq+mn+pq}{2npq+mq-np-2pq-m+p}\right)$.
Consider the determinant

$\Delta = \begin{vmatrix} x_{B_2} - x_{A_2} & x_{C_2} - x_{A_2} \\ y_{B_2} - y_{A_2} & y_{C_2} - y_{A_2} \end{vmatrix} = (x_{B_2} - x_{A_2})(y_{C_2} - y_{A_2}) - (y_{B_2} - y_{A_2}).(x_{C_2} - x_{A_2}).$
With a small help from Maple XVIII, we find the result:

$$\Delta = 0.$$

Thus, $A_2,\ B_2,\ C_2$ are collinear.

3. The Projective model of the Affine Space

Using the projective model of the affine space is a method to create new problems. From the projective problem, we choose different lines at infinity than we obtain different affine problems that do not need to prove. [3]
If we choose the line d at infinity passing through two points $B,\ C$ then two lines AB and EP are parallel and two lines FP and AE are also parallel. The quadrilateral $AEPF$ is a parallelogram. We obtain the following problem in the affine geometry.

Theorem 3.1. *Given two rays Ax and Ay. Let E and F be points on $Ax,\ Ay$, respectively. Construct the parallelogram $AEPF$. Ez and Ft are parallel to AP. An arbitrary line d meets $Ax,\ Ay$ at $B_1,\ C_1$, respectively. Through points $B_1,\ C_1$ draw lines that parallel to $Ay,\ Ax$ and meet Ft at B_2 and Ez at C_2, respectively. Through the point A draw line that is parallel to B_1C_1 and meet the line EF at A_2. Prove that $A_2,\ B_2,\ C_2$ are collinear.*

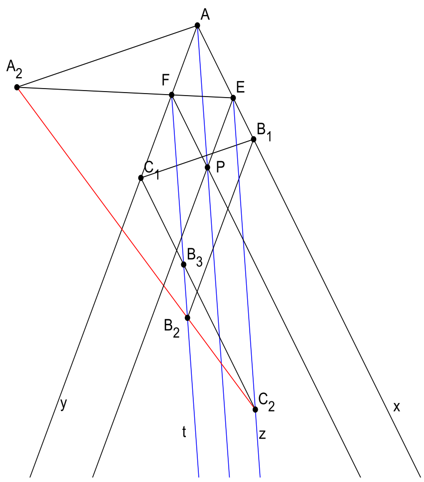

FIGURE 3. The projective model of the affine space

We can prove the theorem directly. We see that, A is on the midline of the trapezoid EC_2B_2F. Thus, the distance from A to EC_2 is equal to the distance from A to B_2F. It follows

$$\frac{EC_2}{FB_2} = \frac{S_{AEC_2}}{S_{AFB_2}} = \frac{S_{AEC_2}}{S_{AB_1C_2}} \cdot \frac{S_{AB_2C_1}}{S_{AFB_2}} \cdot \frac{S_{AB_1C_2}}{S_{AB_2C_1}} = \frac{AE}{AB_1} \cdot \frac{AC_1}{AF}\ (S_{AC_2B_1} = S_{AC_1B_1} = S_{AB_2C_1})\ (1).$$

We have :

$$\frac{AE}{AB_1} \cdot \frac{AC_1}{AF} = \frac{S_{A_2AE}}{S_{A_2AB_1}} \cdot \frac{S_{A_2AC_1}}{S_{A_2AF}} = \frac{S_{A_2AE}}{S_{A_2AF}} = \frac{A_2E}{A_2F}\ (S_{A_2AB_1} = S_{A_2AC_1})\ (2).$$

Since (1), (2) and the converse part of Thales theorem, $A_2,\ B_2,\ C_2$ are collinear. [4]

If we choose the line d at infinity passing through the point A_1 and not passing through the other given points then the quadrilateral BC_1B_1C is a trapezium of the affine space. We obtain the following problem in the affine geometry

Theorem 3.2. *Given a triangle ABC. Let the line d parallels to the line BC and meets AB, AC at C_1, B_1, respectively. GEF is a cevian triangle of the triangle ABC ($G \in BC$, $E \in CA$, $F \in AB$). $GE \cap CC_1 = C_2$; $GF \cap BB_1 = B_2$. Let the line passing through A parallels to BC and meets EF at A_2. Prove that $A_2,\ B_2,\ C_2$ are collinear.*

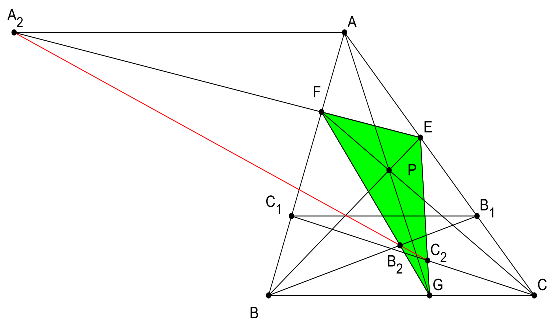

FIGURE 4. The projective model of the affine space

The direct proof of the theorem is as follows
We have

$$\frac{A_2E}{A_2F} = \frac{AE}{AF} \cdot \frac{\sin\widehat{A_2AE}}{\sin\widehat{A_2AF}} = \frac{AE}{AF} \cdot \frac{\sin\widehat{ACB}}{\sin\widehat{ABC}} = \frac{AE}{AF} \cdot \frac{AB}{AC}.$$

Similarly,

$$\frac{B_2F}{B_2G} = \frac{BF}{BG} \cdot \frac{\sin\widehat{ABB_1}}{\sin\widehat{CBB_1}} = \frac{BF}{BG} \cdot \frac{AB_1}{CB_1} \cdot \frac{BC}{BA}.$$
$$\frac{C_2G}{C_2E} = \frac{CG}{CE} \cdot \frac{\sin\widehat{BCC_1}}{\sin\widehat{ACC_1}} = \frac{CG}{CE} \cdot \frac{BC_1}{AC_1} \cdot \frac{AC}{BC}.$$

On the other hand,

$$\frac{AB_1}{CB_1} = \frac{AC_1}{BC_1},\ \frac{AE}{CE} \cdot \frac{CG}{BG} \cdot \frac{BF}{AF} = 1.$$

Thus,

$$\frac{A_2E}{A_2F} \cdot \frac{B_2F}{B_2G} \cdot \frac{C_2G}{C_2E} = 1.$$

By Menelaus theorem, we have that $A_2,\ B_2,\ C_2$ are collinear. [4]

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