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An Another Proof of Dao's Theorem and its Converses

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Abstract. In this article, we give an another synthetic proof of the Dao's generalization of the Simson-Wallace theorem and its converses.

Keywords. Dao's theorem, converses, proof, Euler line, Euler circle, orthopole, orientated angle, vector.

1. Introduction

There is a famous theorem in the classical geometry which is the Simson-Wallace one: *Given a triangle ABC and a point P lying on the circumcircle of this triangle, then three projections from P into BC, CA and AB are collinear, respectively.* [1]. Furthermore, the properties of the Simson-Wallace line can be found in [2], [3], [4], [5] and other documents. In 2014, Dao Thanh Oai who is a Vietnamese engineering stated a nice generalization of the Simson-Wallace line at [6] and it is proved in [7], [8] and [9]. The Dao's theorem is stated as follows:

Theorem 1.1. ([6]). *Given a triangle ABC, P is a point lying on the circumcircle (O) and H is the orthocenter of this triangle. Line ℓ passing through the center of the circumcircle O meets AP, BP, CP at A_P, B_P, C_P, respectively. Let A_0, B_0, C_0 be the projections from A_P, B_P, C_P into $\underline{BC, CA}, AB$, respectively. Then three points A_P, B_P, C_P are collinear, and line $\overline{A_PB_PC_P}$ passes through the midpoint of PH.*

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In this paper, we will define that $d_{P/\ell,ABC} := \overline{A_P B_P C_P}$ is the Dao's line of point P with respect to the line ℓ and triangle ABC, and P is the anti-Dao point of $d_{P/\ell,ABC}$ with respect to ℓ and triangle ABC. By this way, then $d_{P/OP,ABC}$ is the Simson-Wallace line of P with respect to triangle ABC.

We will introduce an another proof of the theorem 1 in the next.

2. An Another Proof of the Theorem 1

Lemma 2.1. *Given a triangle ABC and line ℓ passing through the circumcenter O of this triangle. Let M_A, M_B, M_C be the midpoints of sides BC, CA, AB; and let M_AN_A, M_BN_B, M_CN_C be the chords of the circumcircle (E) of triangle $M_AM_BM_C$ that are parallel to ℓ (can be coincident with ℓ). Then the lines passing through N_A and perpendicular to BC, passing through N_B and perpendicular to CA, passing through N_C and perpendicular to AB meet at a point (Denote by N) lying on (E). Furthermore, N is the orthopole of ℓ with respect to triangle ABC.*

Proof. Let G, H be the centroid and orthocenter of triangle ABC; N_AN is the chord of (E) such that $N_AN \perp BC$; M is the midpoint of AH; and K_A is the point of intersection of N_AN and ℓ (see figure 1).

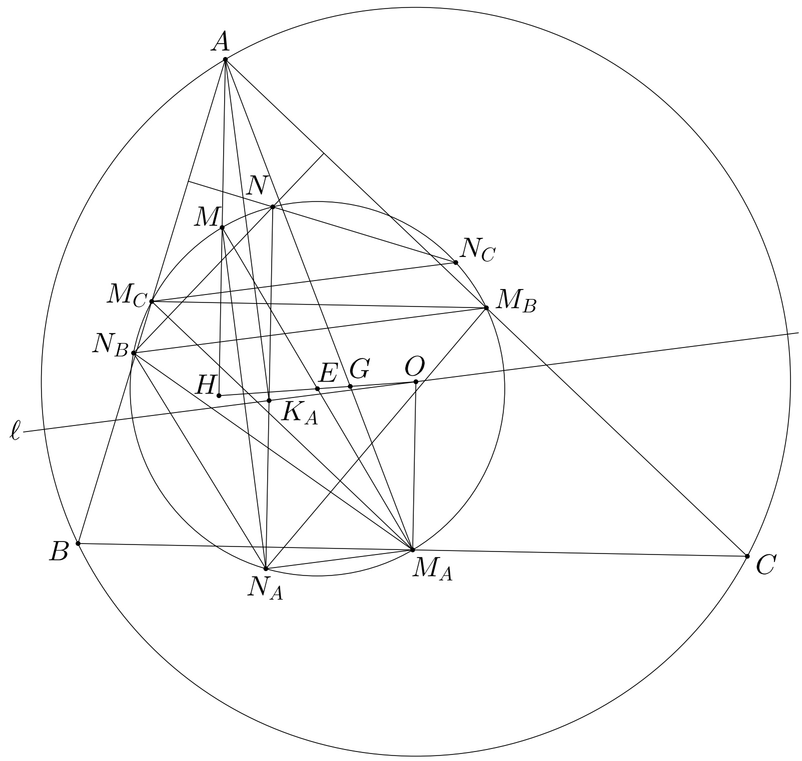

Figure 1.

Using the orientated angle between two lines with the note that M_AN_A is parallel to or coincident with M_BN_B , $M_CM_B \parallel CB$, $M_CM_A \parallel CA$ and $NN_A \perp BC$, we have

$$\begin{aligned}(NN_B, CA) &\equiv (NN_B, NN_A) + (NN_A, CB) + (CB, CA)\\ &\equiv (M_AN_B, M_AN_A) + \frac{\pi}{2} + (M_CM_B, M_CM_A)\\ &\equiv (N_BM_A, N_BM_B) + \frac{\pi}{2} + (M_CM_B, M_CM_A)\\ &\equiv (M_CM_A, M_CM_B) + \frac{\pi}{2} + (M_CM_B, M_CM_A)\\ &\equiv \frac{\pi}{2}\ (mod\,\pi).\end{aligned}$$

If follows $NN_B \perp CA$.

Similarly, $NN_C \perp AB$.

We show that N is the orthopole of ℓ with respect to triangle ABC in the next.

Indeed, since $OM_A \parallel AH(\perp BC)$, the Thales's theorem and the properties of Euler line, we have $\frac{\overline{OM_A}}{\overline{HA}} = \frac{\overline{GO}}{\overline{GH}} = -\frac{1}{2}$. From that, it follows $\overrightarrow{AH} = 2\overrightarrow{OM_A}$.

Since M is the midpoint of AH and note that OM_A is parallel to or coincident with K_AN_A, OK_A is parallel to or coincident with M_AN_A. It follows $\overrightarrow{AM} = \overrightarrow{OM_A} = \overrightarrow{K_AN_A}$. Hence $\overrightarrow{AK_A} = \overrightarrow{MN_A}$.

On the other hand, since MM_A is the diameter of (E), $MN_A \perp M_AN_A$. Since M_AN_A is parallel to or coincident with ℓ. It follows $AK_A \perp \ell$.
This thing means that N lies on the line perpendicular to BC and passing through the projection from A into ℓ.

Similarly, N also lies on the line perpendicular to CA and passing through the projection from B into ℓ. Hence N is the orthopole of ℓ with respect to triangle ABC.

Lemma 2 is proved.

Proof of theorem 1. Let M_A, M, K, Q be the midpoints of BC, HA, HP, AP, respectively; (E) is the Euler circle of triangle ABC; N is the orthopole of ℓ with respect with triangle ABC; M_AN_A is the chord of (E) such that M_AN_A is parallel to or coincident with ℓ; R is the point of intersection of MK and M_AN_A (See figure 2).

We have $\overrightarrow{QK} = \frac{\overrightarrow{AH}}{2} = \overrightarrow{OM_A}$. It follows $\overrightarrow{OQ} = \overrightarrow{M_AK}$. Combining with $OQ \perp AP$ and MK is parallel to or coincident with AP, we follows $M_AK \perp MK$. Since MM_A is the diameter of (E), K belongs to (E).

We easily see that two triangles OQA_P and M_AKR such that their pairs of sides are parallel or coincident each other so there exits either a homothety or a translation that transforms O, Q, A_P into M_A, K, R. Note that $\overrightarrow{OM_A} = \overrightarrow{QK}$. It follows $\overrightarrow{OM_A} = \overrightarrow{QK} = \overrightarrow{A_PR}$. Thus, $A_PR \perp BC \perp A_PA_0$. This thing proves that A_0 belongs to A_PR.
From that $\angle RA_0M_A = \angle RKM_A = \frac{\pi}{2}$. It follows that four points A_0, K, R, M_A are concyclic.

On the other hand, since the lemma 2, $N_AN \perp BC$ so MH_A is parallel to or coincident with NN_A.

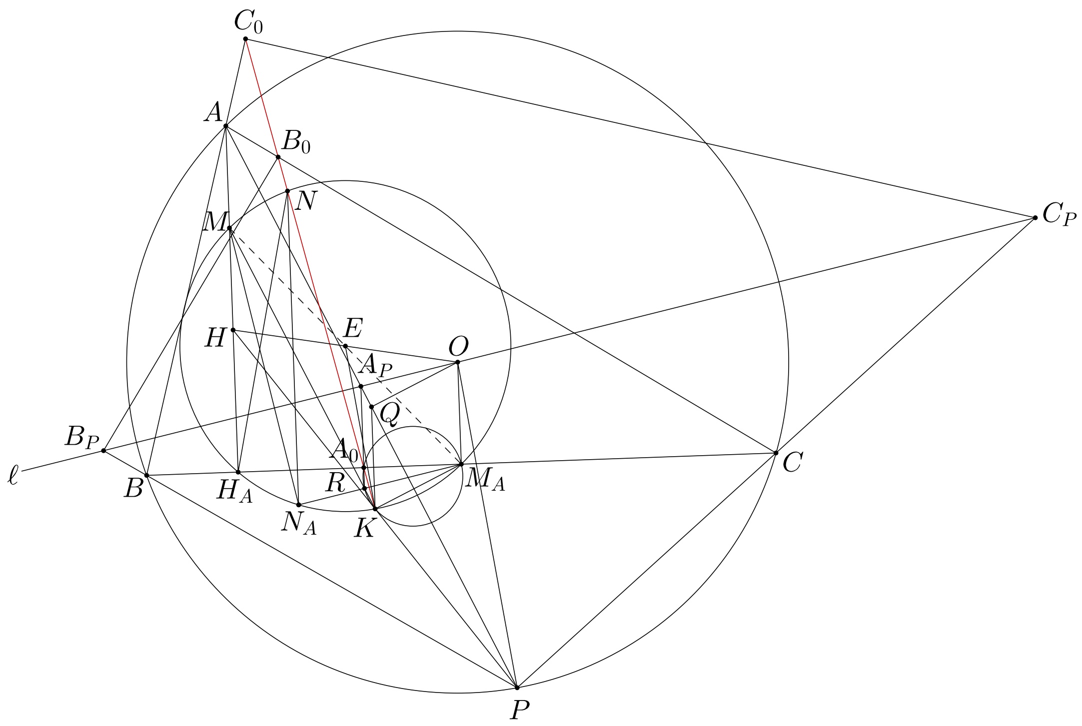

Figure 2.

Hence,

$$\begin{aligned}(KR, KA_0) &\equiv (M_AR, M_AA_0)\\ &\equiv (M_AN_A, M_AH_A)\\ &\equiv (NN_A, NH_A)\\ &\equiv (MN_A, MH_A)\\ &\equiv (N_AM, N_AN)\\ &\equiv (KM, KN)\\ &\equiv (KR, KN)\ (mod\ \pi).\end{aligned}$$

It follows that KA_0 and KN are coincident. The corollary is that A_0 belongs to KN.

Similarly, we have B_0 and C_0 belonging to KN.

Hence, four points A_0, B_0, C_0 and K are collinear. This means that theorem 1 is proved.

Remark. *Line $d_{P/\ell,ABC}$ passes through the orthopole of line ℓ with respect to triangle ABC.*

3. Two Converses of Theorem 1 and their Proofs

Theorem 3.1. ([10]). *Given a triangle ABC, P is a point lying on the circumcircle (O) and H is the orthocenter of this triangle. Line d passing through the midpoint of PH meets BC, CA, AB at A_0, B_0, C_0, respectively. Line passing through A_0 perpendicular to BC meets PA at A_P, line passing through B_0 perpendicular* *to CA meets PB at B_P, and line passing through C_0 perpendicular to AB meets PC at C_P. Then A_P, B_P, C_P and the circumcenter O are collinear.*

Proof. Let (E) be the Euler circle of triangle ABC; let M_A, M_b, M_C be the midpoints of sides BC, CA, AB; NN_A, NN_B, NN_C are the chords of (E) that are perpendicular to BC, CA, AB, respectively. Let K be the midpoint of HP (see figure 3).

Since E, K are the midpoints of HO, HP, respectively, $\overrightarrow{EK} = \frac{\overrightarrow{OP}}{2}$. This thing proves that K belongs to the circle (E). Suppose that d meets (E) at K and N (N can be coincident with K in the case that d is tangent with (E)).

Let M be the midpoint of AH; R is the point of intersection of MK and M_AN_A, H_A is the projection from A into BC.

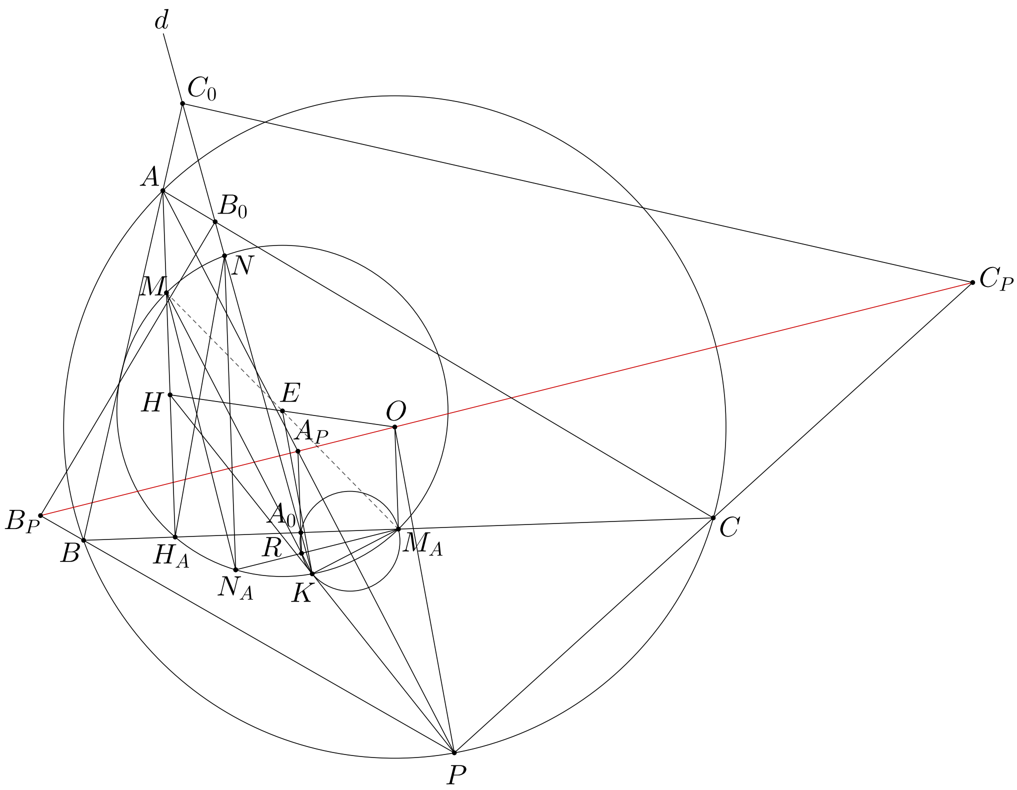

Figure 3.

We have

$$\begin{aligned}(KR, KA_0) &\equiv (KM, KN)\\ &\equiv (N_AM, N_AN)\\ &\equiv (H_AM, H_AN)\\ &\equiv (NN_A, NH_A)\\ &\equiv (M_AN_A, M_AH_A)\\ &\equiv (M_AR, M_AA_0)\ (mod\,\pi).\end{aligned}$$

It follows that four points A_0, K, M_A, R are concyclic. Note that MM_A is the diameter of (E) so $\angle RKM_A = \frac{\pi}{2}$. Hence $\angle RA_0M_A = \frac{\pi}{2}$. This thing proves that A_0 belongs A_PR, and A_PR is parallel to or coincident with AH. Since MK is parallel to or coincident with AA_P, it follows that $\overrightarrow{A_PR} = \overrightarrow{AM} = \overrightarrow{OM_A}$. From that $\overrightarrow{OA_P} = \overrightarrow{M_AR}$, it follows that OA_P are parallel to or coincident with M_AN_A.

Similarly, we also have that OB_P is parallel to or coincident with M_BN_B and OC_P is parallel to or coincident with M_CN_C.

Since the lemma 2, we easily see that pairs of M_AN_A, M_BN_B, M_CN_C are parallel or coincident each other. From that, it follows that OA_P, OB_P, OC_P are coincident or four points A_P, B_P, C_P and O are collinear.

Theorem 4 is proved.

Remark. *Theorem 4 gives us the way of definition of ℓ when we know Dao line and its anti-Dao point.*

Theorem 3.2. *Given a triangle ABC inscribed in a circle (O). A line ℓ passes through the circumcenter O. Line d passing through the orthopole of line ℓ with respect to triangle ABC and meets BC, CA, AB at A_0, B_0, C_0, respectively. Lines passing through A_0 perpendicular to BC, passing through B_0 perpendicular to CA and passing through C_0 perpendicular to AB meet ℓ at A', B' and C', respectively. Then lines $A'A, B'B$ and $C'C$ meet at a point lying on the circle (O).*

Proof. (see figure 4). Let N be the orthopole of ℓ with respect to triangle ABC; (E) is the Euler circle of triangle ABC. By the lemma 2, N belongs to the circle (E).

Let H be the orthocenter of triangle ABC; d meets (E) at N and K; P is the symmetric point of H under a symmetry about center K. Since $\overrightarrow{OP} = 2\overrightarrow{EK}$, it follows that P belongs (O).

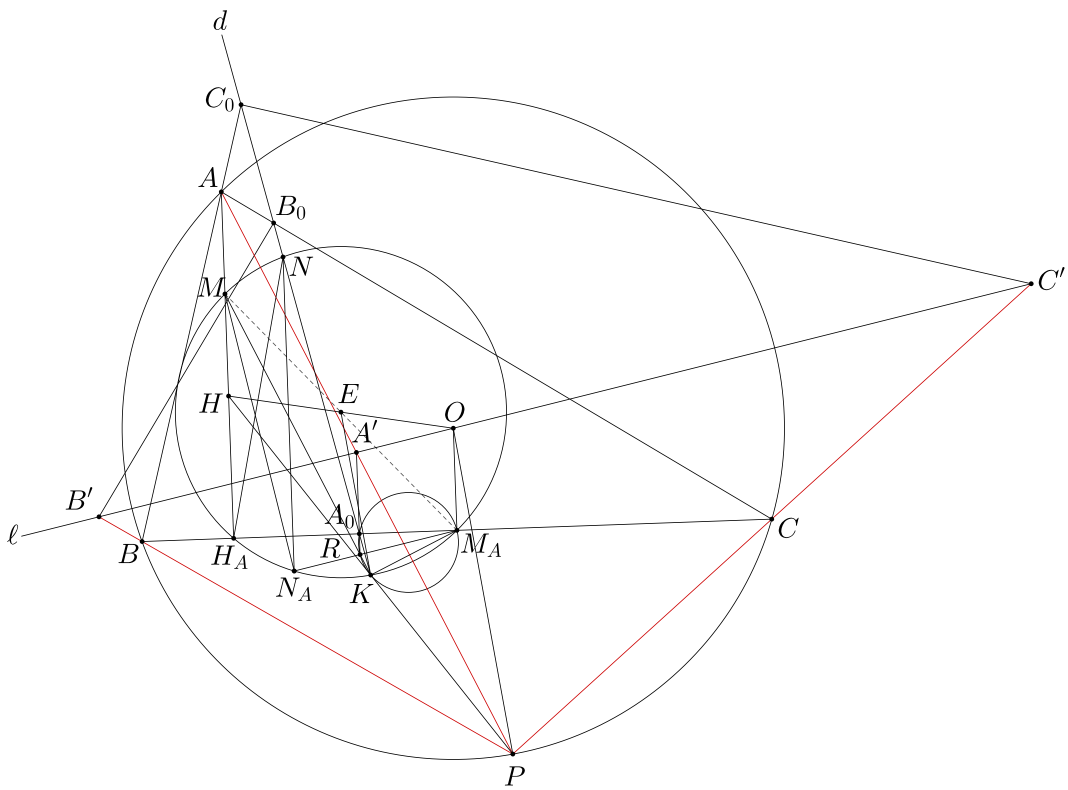

Figure 4.

Let M, M_A be the midpoints of AH, BC, respectively then MM_A is the diameter of (E). Construct the chord M_AN_A of (E) such that M_AN_A is parallel to or coincident with ℓ; M_AN_A meets MK at R; and H_A is the projection from A into BC. We have

$$\begin{aligned}(M_AR, MA_0) &\equiv (M_AN_A, M_AH_A)\\ &\equiv (NN_A, NH_A)\\ &\equiv (MN_A, MH_A)\\ &\equiv (N_AM, N_AN)\\ &\equiv (KM, KN)\\ &\equiv (KR, KA_0)\ (mod\,\pi).\end{aligned}$$

Thus, four points A_0, K, M_A, R is concyclic. Since $M_AK \perp RK$, $RA_0 \perp M_AA_0 \equiv BC$. It follows that three points A', A_0, R are collinear and $A'R$ is parallel or coincident with OM_A. We also have that OA' is parallel to or coincident with M_AR so $\overrightarrow{A'R} = \overrightarrow{OM_A} = \overrightarrow{AM}$. It follows $\overrightarrow{AA'} = \overrightarrow{MR}$. Because $MR \equiv MK$ and $\overrightarrow{MK} = \frac{\overrightarrow{AP}}{2}$, two vectors $\overrightarrow{AA'}$ and $\overrightarrow{AP}$ are parallel. Thus, three points A, A', P are collinear or AA' passes through P.

Similarly, BB' and CC' also pass through P.

Hence, theorem 6 is proved.

Remark. *Theorem 6 gives us the way of the definition of anti-Dao point when we know Dao line and ℓ.*

REFERENCES

[1] H. S. M. Coxeter and S.L. Greitzer, Geometry revisited, Math. Assoc. America, 1967: p.41.
[2] Ramler, O. J. The Orthopole Loci of Some One-Parameter Systems of Lines Referred to a Fixed Triangle. Amer. Math. Monthly 37, 130-136, 1930.
[3] Butchart, J. H. The Deltoid Regarded as the Envelope of Simson Lines. Amer. Math. Monthly 46, 85-86, 1939.
[4] van Horn, C. E. The Simson Quartic of a Triangle. Amer. Math. Monthly 45, 434-437, 1938.
[5] Gallatly, W. The Simson Line. Ch. 4 in The Modern Geometry of the Triangle, 2nd ed. London: Hodgson, pp. 24-36, 1913.
[6] T. O. Dao, Advanced Plane Geometry, message 1781, September 20, 2014.
[7] Leo Giugiuc, A proof of Dao's generalization of the Simson line theorem, Global Journal of Advanced Research on Classical and Modern Geometries, Vol.5, (2016), Issue 1, page 30-32.
[8] T. L. Tran, Another synthetic proof of Dao's generalization of Simson line theorem and its converse, Global Journal of Advanced Research on Classical and Modern Geometries, Vol.5, (2016), Issue 2, pp.89-92.
[9] V. L. Nguyen, Another synthetic proof of Dao's generalization of the Simson line theorem, Forum Geometricorum, 16 (2016) 57–61.
[10] https://www.artofproblemsolving.com/community/c6h1212168p6012903.