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Another Generalization of the Sawayama and Thébault's Theorem

Dao Thanh Oai
Cao Mai Doai, Quang Trung, Kien Xuong, Thai Binh, Vietnam
e-mail: daothanhoai@hotmail.com

Abstract. In this note we propose another generalization of the Sawayama lemma and Sawayama-Thébault's Theorem.

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1. Introduction

The famous Sawayama-Thébault's theorem is as follows:

Theorem 1.1 (Sawayama-Thébault). *Let ABC be a triangle with the incenter I, let D be a point on the side BC. Let O_1 be the center of the circle tangent to the circumcircle and segments AD and BD, (O_2) be the center of the circle tangent to the circumcircle and segments AD and CD. Then O_1, O_2 and I are collinear (Figure 1)*

The Sawayama-Thébault's theorem is problem 3887 in the American Mathematical Monthly, the problem was proposed by V. Thébault (1938) [1] and apparently remained unsolved until the 1970s. It was therefore a surprise when Ayme (2003) discovered that in fact the problem was known and solved by Sawayama (1905) [3], so now we call the theorem is Sawayama-Thébault's Theorem [2]. The lemma to directly solve the theorem now we call is Sawayama lemma.

Lemma 1.1 (Sawayama). *Let ABC be a triangle with the incenter I, let D be a point on the side BC. Let (w) be the circle such that (w) tangent to the circumcircle, and (w) tanegnt to segment DC at E and (w) tangent to segment AD at F, then E, F and I are collinear (Figure 2)*

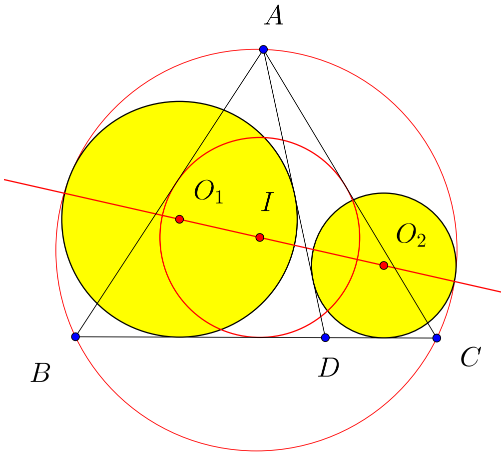

FIGURE 1. Sawayama-Thébault's Theorem

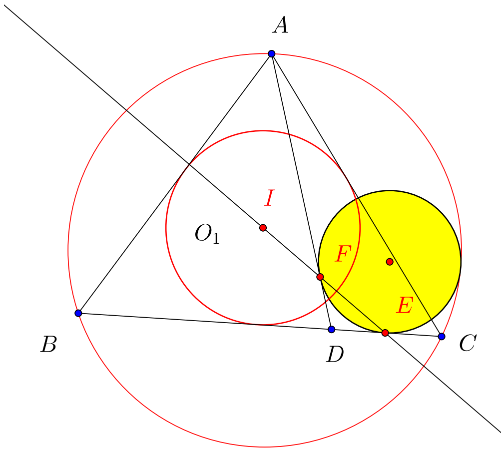

FIGURE 2. Sawayama lemma

In [5] the author proposed a generalization of the Sawayama-Thébault's theorem associated with a circumconic through two points isogonal conjugate. In this note, we proposed another generalization of the Sawayama lemma and Sawayama-Thébault's Theorem associated arbitrary circle thought two vertices.

2. Another generalization of the Sawayama lemma

Problem 2.1. *Let ABC be a triangle with the incenter I, let (O) be a circle through B, C. Let (O_A) be a circle such that (O_A) tangent to AB, AC, and tangent to (O), such that common point of (O_A), (O) and A are in the same half plane divides by BC, and A, O_A, I are collinear. Let P be a point outside of (O_A), let L be a line through P and tangent to (O_A). Let (O_1) be the circle tangent to BC, and tangent to L, and (O_1) tangent to (O) such that:*

1-if (O_A) externally tangent to (O), selected (O_1) and (O_A) are not in the same half plane divides by L, (Figure 3).

2-if (O_A) internally tangent to (O), selected (O_1) and (O_A) are in the same half plane divides by L, (Figure 4).

Let (O_1) tangent to BC at D, (O_1) tangent to L at E. Then show that D, E, I are collinear.

3. Another generalization of the Sawayama-Thebault theorem

Problem 3.1. *Let ABC be a triangle with the incenter I, let (O) be a circle through B, C. Let (O_A) be a circle such that (O_A) tangent to AB, AC, and (O), such that common point of (O_A), (O) and A are in the same half plane divides by BC, and A, O_A, I collinear. Let P be a point outside of (O_A), let L_1, L_2 be two lines through P and tangent to (O_A). Let (O_1), (O_2) be two circles, such that (O_1) tangent to (O), (O_1) tangent to L_1 and (O_1) tangent to BC, (O_2) tangent to (O), (O_2) tangent to L_2 and (O_2) tangent to BC.*

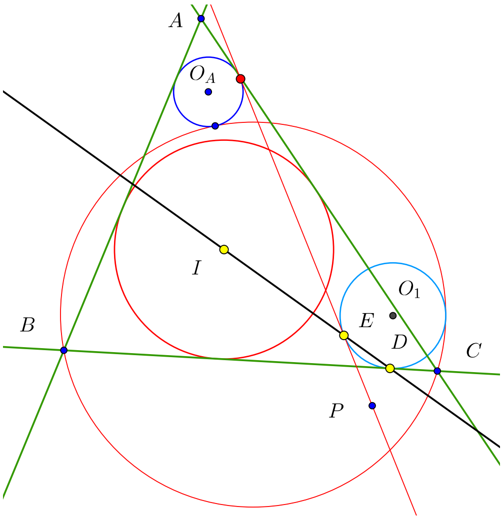

FIGURE 3. (O_A) externally tangent to (O)

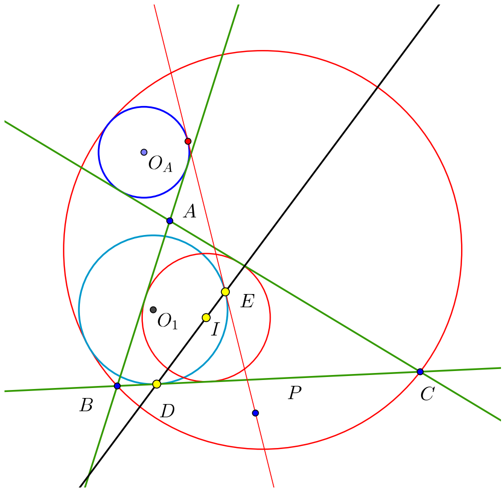

FIGURE 4. (O_A) internally tangent to (O)

1. If (O_A) externally tangent to (O), selected $(O_1),(O_2)$, such that (O_1) and (O_A) are not in the same half plane divides by L_1, (O_2) and (O_A) are not the same half plane divides by L_2 (Figure 5).

2. If (O_A) internally tangent to (O), selected $(O_1),(O_2)$ such that (O_1) and (O_A) are the same half plane divides by L_1, (O_2) and (O_A) are the same half plane divides by L_2 (Figure 6).

Then show that the line O_1O_2 through a fixed point when P move on a given line.

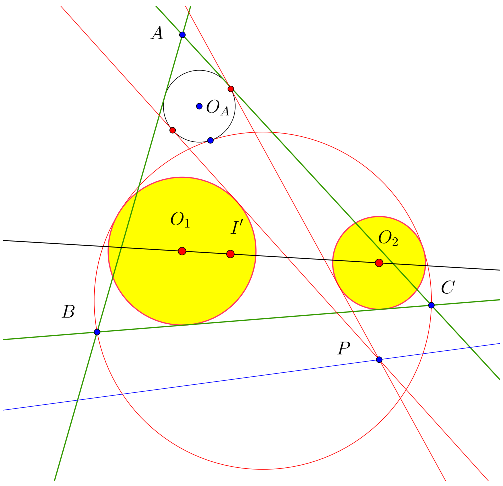

FIGURE 5. (O_A) externally tangent to (O)

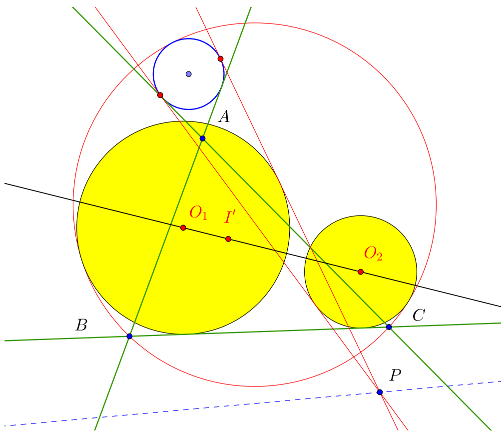

FIGURE 6. (O_A) internally tangent to (O)

4. VARIANTS

There are many variants of problem 1 and problem 2. Example:

Problem 4.1. *Let ABC be a triangle with the incenter I, and excenter E_A, let (O) be a circle through B, C. Let (O_A) be a circle such that (O_A) tangent to AB,*

AC, and (O_A) internally (or externally) tangent to (O). Let common point of two circles $(O_A), (O)$ and A are not in the same half plane divides by BC. Let P be a point outside of (O_A), let L be a line through P and tangent to (O_A). Let (O_1) be the circle such that (O_1) and (O_A) are not in the same half plane divides by L(or (O_1) and (O_A) are in the same half plane divides by L if (O_A) externally tangent to (O)), and (O_1) tangent to BC, and (O_1) tangent to L. Let (O_1) tangent with BC at D, (O_1) tangent L at E. Then show that:

1-D, E, I are collinear if (O_A) internally tangent to (O) (Figure 7).

2-D, E, E_A are collinear (O_A) externally tangent to (O) (Figure 8).

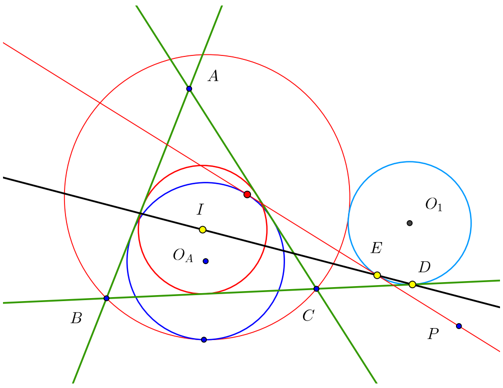

FIGURE 7. (O_A) externally tangent to (O)

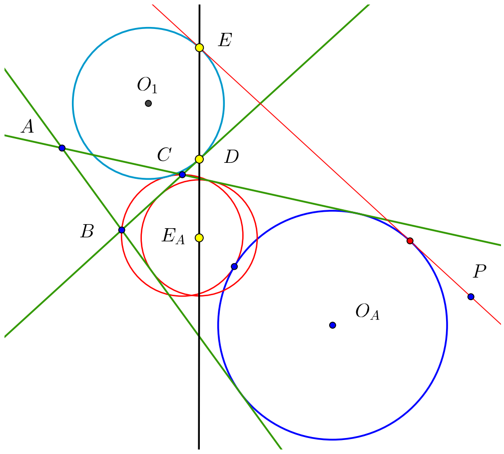

FIGURE 8. (O_A) internally tangent to (O)

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