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Computer Discovered Mathematics: Half-Cevian Triangles

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Abstract. We present results about the Half-Cevian Triangles discovered by the computer program "Discoverer".

Keywords. half-cevian triangle, triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

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1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [3].

In this paper we present theorems about Half-Cevian Triangles discovered by the computer program "Discoverer".

Let P be an arbitrary point in the plane of triangle ABC. Denote by $PaPbPc$ the cevian triangle of point P. Let Ha be the midpoint of the cevian APa and define Hb and Hc similarly. We call triangle $HaHbHc$ as the *Half-Cevian Triangle of Point P*.

Figure 1 illustrates the definition. In Figure 1 P is an arbitrary point and $HaHbHc$ is the Half-Cevian Triangle of P.

Note that the Half-Cevian Triangle of the Orthocenter is the known Half-Altitude Triangle. See [11, Half-Altitude Triangle].

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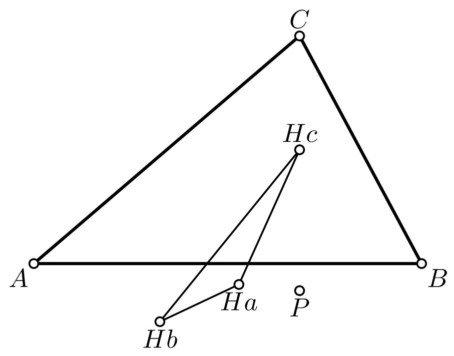

FIGURE 1.

2. PRELIMINARIES

In this section we review some basic facts about barycentric coordinates. We refer the reader to [12],[1],[8],[4],[5],[10]. The labeling of triangle centers follows Kimberling's ETC [7]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [11].

The reference triangle ABC has vertices $A = (1, 0, 0), B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

$$\text{For all } k \in \mathbb{R} - \{0\} : \; P = (u, v, w) \text{ means that } P = (u, v, w) = (ku, kv, kw).$$

Given a point $P(u, v, w)$. Then P is *finite*, if $u + v + w \neq 0$. A finite point P is *normalized*, if $u + v + w = 1$. A finite point could be put in normalized form as follows: $P = (\frac{u}{s}, \frac{v}{s}, \frac{w}{s})$,where $s = u + v + w$.

Three points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ lie on the same line if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0. \tag{1}$$

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{2}$$

The intersection of two lines $L_1 : \; p_1x + q_1y + r_1z = 0$ and $L_2 : \; p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{3}$$

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{4}$$

If the barycentric coordinates of points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ are normalized, then the area of $\triangle P_1P_2P_3$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta. \tag{5}$$

where Δ is the area of the reference triangle ABC.

Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates, the midpoint M of P and Q is as follows:

$$M = \left(\frac{u_1 + u_2}{2}, \frac{v_1 + v_2}{2}, \frac{w_1 + w_2}{2}\right). \tag{6}$$

Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates, the reflection R of P in Q is as follows:

$$R = (2u_2 - u_1, 2v_2 - v_1, 2w_2 - w_1). \tag{7}$$

Given a point $P(u, v, w)$, the complement of P is the point $(v + w, w + u, u + v)$, the anticomplement of P is the point $(-u + v + w, -v + w + u, -w + u + v)$, the isotomic conjugate of P is the point (vw, wu, uv), and the isogonal conjugate of P is the point (a^2vw, b^2wu, c^2uv).

3. Half-Cevian Triangle of a Point P

Theorem 3.1. *The barycentric coordinates of the half-cevian triangle $HaHbHc$ of a point $P = (u, v, w)$ are as follows:*

$$Ha = (v + w, v, w),\ Hb = (u, u + w, w),\ Hc = (u, v, u + v).$$

Proof. The Cevian triangle $PaPbPc$ of a point $P = (u, v, w)$ is as follows:

$$Pa = (0, v, w),\ Pb = (u, 0, w),\ Pc = (u, v, 0)$$

We put the points Pa, Pb and Pc in normalized form. By using (6) we find the midpoint Ha of segment APa as follows: $Ha = (v + w, v, w)$. Similarly, we find the midpoints of the segments BPb and CPc respectively as follows: $Hb = (u, u + w, w)$ and $Hc = (u, v, u + w)$. □

Theorem 3.2. *The area of the Half-Cevian Triangle of Point $P = (u, v, w)$ is as follows:*

$$area(P) = \frac{2\,uvw\,\Delta}{(u + v)(v + w)(w + u)}.$$

where Δ is the area of triangle ABC.

Proof. See (5) and Theorem 3.1. □

The following Theorem has barycentric proof in [8, §38], [9, §25.3] and synthetic proof in [6, Problem 1030].

Theorem 3.3. *The Half-Cevian Triangle of an arbitrary point P and the Medial Triangle are perspective. The Perspector is the Complement of the Isotomic Conjugate of point P.*

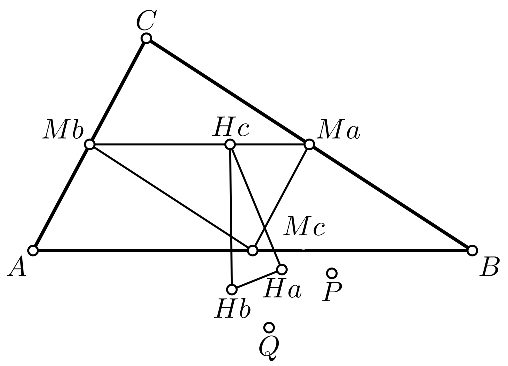

FIGURE 2.

Figure 2 illustrates Theorem 3.3. In Figure 2, P is an arbitrary point, $HaHbHc$ is the Half-Cevian Triangle of P, $MaMbMc$ is the Medial Triangle of triangle ABC, and Q is the Complement of the Isotomic Conjugate of point P. At the same time, Q is the intersection point of lines $MaHa$, $MbHb$ and $McHc$. These lines are not drown in the figure.

Theorem 3.4. *The Half-Cevian Triangle of a Point P and the Triangle of Reflections of the Vertices of Triangle ABC in the Complement of the Isotomic Conjugate of Point P are perspective. The barycentric coordinates of the Perspector $Q = (uQ, vQ, wQ)$ are as follows:*

$$uQ = u(6vw^2u+6v^2wu+v^2w^2+vw^3+4u^2vw+2u^2v^2+v^3u+v^3w+2w^2u^2+w^3u),$$
$$vQ = v(6wu^2v+6w^2uv+w^2u^2+wu^3+4v^2wu+2v^2w^2+w^3v+w^3u+2u^2v^2+u^3v),$$
$$wQ = w(6uv^2w+6u^2vw+u^2v^2+uv^3+4w^2uv+2w^2u^2+u^3w+u^3v+2v^2w^2+v^3w).$$

Proof. Given triangle ABC and a point $P = (u, v, w)$. The barycentric coordinates of the Half-Cevian Triangle of P are given in Theorem 3.1. We find the barycentric coordinates of the Complement of the Isotomic Conjugate point P. Label it R. Then we have

$$R = (u(v + w), v(w + u), w(u + v)).$$

Now by using (7) we find the barycentric coordinates of the reflections of points A, B and C in point R. Denote the reflections by Ra, Rb and Rc, respectively. We obtain

$$Ra = (-vw, v(u + w), w(u + v)),$$
$$Rb = (u(v + w), -uw, w(u + v)),$$
$$Rc = (u(v + w), v(u + w), -uv).$$

By using (2) we find the barycentric equation of lines $L_1 = HaRa, L_2 = HbRb$ and $L_3 = HcRc$ as follows:

$$L_1 : \ vw(v - w)x - w(2vw + uv + uw + v^2)y + v(2vw + uv + uw + w^2)z,$$
$$L_2 : \ w(2uw + uv + vw + u^2)x - uw(u - w)y - u(2uw + uv + vw + w^2)z,$$
$$L_3 : \ v(2uv + uw + vw + u^2)x - u(2uv + uw + vw + v^2)y - uv(u - v)z.$$

By using (4) we verify that the lines L_1, L_2 and L_3 concur in a point. By using (3) we find the intersection of lines L_1 and L_2. The intersection is the perspector of triangles $HaHbHc$ and $RaRbRc$. The barycentric coordinates of the perspector are given in the statement of the theorem. □

4. Problems for the Reader

The "Discoverer" has discovered a number of theorems about the Half-Cevian Triangle of an arbitrary point P. Below we give a few theorems as problems for the Reader.

Problem 4.1. *Prove that the Half-Cevian Triangle of Point P and the Triangle of Reflections of the Vertices of Triangle ABC in the Complement of the Complement of the Complement of Point P are perspective. Find the barycentric coordinates of the Perspector.*

Problem 4.2. *Prove that the Half-Cevian Triangle of Point P and the Triangle of Reflections of the Vertices of the Cevian Triangle of the Isotomic Conjugate of Point P in the Point P are perspective. Prove that the Perspector is the Anticomplement of Point P.*

Problem 4.3. *Prove that the Half-Cevian Triangle of Point P and the Euler Triangle of the Complement of the Isotomic Conjugate of Point P are perspective. Find the barycentric coordinates of the Perspector.*

The Folder 1 of the enclosed Supplementary Material contains 224 theorems about perspective half-cevian triangles of triangle centers. All theorems are discovered by the "Discoverer". These theorems could be considered as problems. There are 105 perspectors which are points available in the Kimberling's ETC [7] and the rest of 139 perspectors are new remarkable points which are not available in [7].

5. Special Case: P = Incenter

From Theorem 3.1 we obtain as a special case the barycentric coordinates of the Half-Cevian Triangle $HaHbHc$ of the Incenter as follows:

$$Ha = (b + c, b, c),\ Hb = (a, a + c, c),\ Hc = (a, b, a + b).$$

From Theorem 3.2 we obtain as a special case the area of the Half-Cevian Triangle of the Incenter as follows:

$$area = \frac{2\,abc\,\Delta}{(a + b)(b + c)(c + a)}.$$

Theorem 5.1. *The Perspector of the Half-Cevian Triangle of the Incenter and the*

1. *Medial Triangle is the Grinberg Point X(37).*
2. *Extouch Triangle is the Spieker Center X(10).*
3. *Intouch Triangle is the point X(226).*

Proof.

(1) The Grinberg Point X(37) is the Complement of the Isotomic Conjugate of the Incenter. We apply Theorem 3.3. □

(2) The Spieker Point has barycentric coordinates $Sp = (b + c, c + a, a + b)$. The Extouch Triangle has barycentric coordinates

$$Ea = (0, c+a-b, a+b-c), Eb = (b+c-a, 0, a+b-c), Ec = (b+c-a, c+a-b, 0).$$

Denote by $HaHbHc$ the Half-Cevian Triangle of the Incenter. We use (1) in order to prove that the points Ha, Sp and Ea lie on the same line. We form the matrix

$$A = \begin{bmatrix} b+c & b & c \\ b+c & c+a & a+b \\ 0 & c+a-b & a+b-c \end{bmatrix}.$$

Since det(A) = 0, by (1) we conclude that the points Ha, Sp and Ea lie on the same line. Similarly, points Hb, Sp and Eb lies on the same line and points Hc, Sp and Ec lie on the same line. Hence, point Sp is the intersection point of lines $HaEa, HbEb$ and $HcEc$. □

6. SPECIAL CASE: P = CENTROID

From Theorem 3.1 we obtain as a special case the barycentric coordinates of the Half-Cevian Triangle $HaHbHc$ of the Centroid as follows:

$$Ha = (2, 1, 1),\ Hb = (1, 2, 1),\ Hc = (1, 1, 2).$$

From Theorem 3.2 we obtain as a special case that the area of the Half-Cevian Triangle of the Centroid is equal to $\frac{\Delta}{16}$.

The "Discoverer" has investigated 195 Remarkable Points of the Half-Cevian Triangle of the Centroid. See the enclosed Folder 2, List P. Of these 91 are available in the Kimberling's ETC [7]. See enclosed Folder 2, List K.

The Table 1 gives a few of the remarkable points of the Half-Cevian Triangle of the Centroid in terms of the points of the Reference triangle.

	Point of the Half-Cevian Triangle of the Centroid	**Point of the Reference Triangle**
1	X(1) Incenter	X(1125)
2	X(2) Centroid	X(2) Centroid
3	X(3) Circumcenter	X(140) Nine-Point Center of the Medial Triangle
4	X(4) Orthocenter	X(5) Nine-Point Center
5	X(5) Nine-Point Center	X(3628)
6	X(6) Symmedian Point	X(3589)
7	X(7) Gergonne Point	X(142) Mittenpunkt of the Medial Triangle
8	X(8) Nagel Point	X(10) Spieker Center
9	X(9) Mittenpunkt	X(6666)
10	X(10) Spieker Center	X(3634)

Table 1

In the enclosed Folder 2, List D, there are 104 Remarkable Points of the Half-Cevian Triangle of the Centroid which are not available in Kimberling's ETC [7]. These points are new remarkable points.

In Table 2, **C** denotes a remarkable circle of the Half-Cevian Triangle of the Centroid.

	Circle C of the Half-Cevian Triangle of the Centroid	**Center of circle C as point of the Reference triangle**
1	Circumcircle	X(140) Nine-Point Center of the Medial Triangle
2	Incircle	X(1125)
3	Nine-Point Circle	X(3628)
4	Excentral Circle	X(6684)
5	Antimedial Circle	X(5) Nine-Point Center
6	Spieker Circle	X(3634)
7	Orthocentroidal Circle	X(547)
8	Moses Circle	X(6683)
9	Inner Johnson-Yff Circle	X(3822)
10	Outer Johnson-Yff Circle	X(3825)
11	Gallatly Circle	X(6683)
12	Cosine Circle	X(3589)

Table 2

7. Special Case: P = Circumcenter

From Theorem 3.1 we obtain as a special case the barycentric coordinates of the Half-Cevian Triangle $HaHbHc$ of the Circumcenter as follows:

$$Ha = (2b^2c^2 + b^2a^2 - b^4 + c^2a^2 - c^4, b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)),$$

$$Hb = (a^2(b^2 + c^2 - a^2), b^2a^2 + 2c^2a^2 - a^4 + b^2c^2 - c^4, c^2(a^2 + b^2 - c^2)),$$

$$Hc = (a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), 2b^2a^2 + c^2a^2 - a^4 + b^2c^2 - b^4).$$

Problem 7.1. *Find the area of the Half-Cevian Triangle of the Circumcenter.*

8. Special Case: P = Orthocenter

This special case is studied in [11, Half-Altitude Triangle].

From Theorem 3.1 we obtain as a special case the barycentric coordinates of the Half-Cevian Triangle $HaHbHc$ of the Orthocenter as follows:

$$Ha = (2a^2, a^2 + b^2 - c^2, c^2 + a^2 - b^2),$$

$$Hb = (a^2 + b^2 - c^2, 2b^2, b^2 + c^2 - a^2),$$

$$Hc = (c^2 + a^2 - b^2, b^2 + c^2 - a^2, 2c^2).$$

From Theorem 3.2 we obtain as a special case the area of the Half-Cevian Triangle of the Orthocenter as follows:

$$area = \frac{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)}{16a^2b^2c^2}\Delta = \frac{cosA\, cosB\, cosC}{2}\Delta.$$

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

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