International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
November 2015, Volume 0, No.0, pp.70-79.
Received 15 September 2015. Published on-line 1 October 2015
web: http://www.journal-1.eu/

Computer Discovered Mathematics: Haimov Triangles

Sava Grozdev[a] and Deko Dekov[b2]
[a] VUZF University of Finance, Business and Entrepreneurship,
Gusla Street 1, 1618 Sofia, Bulgaria
e-mail: sava.grozdev@gmail.com
[b]Zahari Knjazheski 81, 6000 Stara Zagora, Bulgaria
e-mail: ddekov@ddekov.eu
web: http://www.ddekov.eu/

Abstract. By using the computer program "Discoverer", we investigate the Haimov triangles. We present theorems about the Haimov triangle of a point P, and we also consider the special case when P is Centroid. Theorems about other special cases, e.g. when P is the Incenter, Circumcenter, Orthocenter, and so on, could be easily discovered by the "Discoverer" upon request. Haim Haimov [9] has suggested the "Discoverer" to investigate Haimov points and to identify them as remarkable points of the triangle. We answer the Haimov's suggestion in Section 5. Haim Haimov [9] also has suggested the "Discoverer" to find triangles which are perspective with the Haimov triangle of the Centroid. We answer this Haimov's suggestion in Section 6.2.

Keywords. Haimov triangle, Haimov point, triangle geometry, remarkable point, computer-discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [3].

In this paper, by using the "Discoverer", we investigate the Haimov triangles. The paper contains selected theorems about Haimov triangles. We expect that the majority of these theorems are new, discovered by a computer. Many of the

[2]Corresponding author

proofs of theorem are not presented here. We recommend to the reader to find the proofs and to submit them for publication.

Recall the definition of the Haimov triangle. See [9]: Let P be a point of Ceva in the plane of $\triangle ABC$. Let $A_1B_1C_1$ be the Cevian triangle of P. Points A, B and B_1 define a circle, and points A, C and C_1 define another circle. Point A is an intersection of these two circles. Label by Qa the second intersection of the circles. Similarly, define points Qb and Qc. Then $\triangle QaQbQc$ is the *Haimov triangle of point P*.

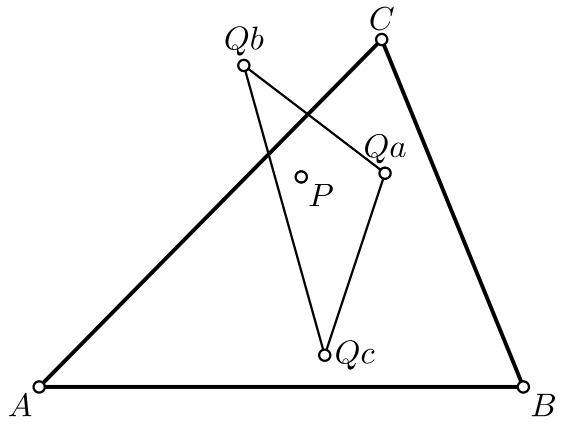

Figure 1.

Figure 1 illustrates the definition of the Haimov triangle of a point P. In figure 1, P is an arbitrary point of Ceva and $QaQbQc$ is the Haimov Triangle of point P.

In this paper we present theorems about the Haimov Triangle of a point P, and we consider the special case when P is the Centroid. Theorems about other special cases, e.g. when P is the Incenter, Circumcenter, Orthocenter, and so on, could be easily discovered by the "Discoverer" upon request.

Haim Haimov [9] has suggested the "Discoverer" to investigate Haimov points and to identify them as remarkable points of the triangle. We answer the Haimov's suggestion in Section 5.

Haim Haimov [9] also has suggested the "Discoverer" to find triangles which are perspective with the Haimov triangle of the Centroid. We answer this Haimov's suggestion in Section 6.2.

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to [19],[2],[1],[13],[7],[8],[12],[15]. The labeling of triangle centers follows Kimberling's ETC [10]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [4], Contents, Definitions, and in [17],[18].

The reference triangle ABC has vertices $A = (1, 0, 0), B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

$$\text{For all } k \in \mathbb{R}\setminus\{0\}: \; P = (u, v, w) \text{ means that } P = (u, v, w) = (ku, kv, kw).$$

Given a point $P(u, v, w)$. Then P is *finite*, if $u + v + w \neq 0$. A finite point P is *normalized*, if $u + v + w = 1$. A finite point could be put in normalized form as follows: $P = (\frac{u}{q}, \frac{v}{q}, \frac{w}{q})$, where $q = u + v + w$. We use the Conway's notation:

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \ S_B = \frac{c^2 + a^2 - b^2}{2}, \ S_C = \frac{a^2 + b^2 - c^2}{2},$$

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{2.1}$$

The intersection of two lines $L_1 : p_1x + q_1y + r_1z = 0$ and $L_2 : p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{2.2}$$

The infinite point of a line $L : px + qy + rz = 0$ is the point (f, g, h), where $f = q - r$, $g = r - p$ and $h = p - q$.

The equation of the line through point $P(u, v, w)$ and parallel to the line $L : px + qy + rz = 0$ is as follows:

$$\begin{vmatrix} f & g & h \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{2.3}$$

The equation of the line through point $P(u, v, w)$ and perpendicular to the line $L : px+qy+rz = 0$ is as follows (The method is discovered by Floor van Lamoen):

$$\begin{vmatrix} F & G & H \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{2.4}$$

where $F = S_Bg - S_Ch$, $G = S_Ch - S_Af$, and $H = S_Af - S_Bg$.

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{2.5}$$

If the barycentric coordinates of points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ are normalized, then the area of $\triangle P_1P_2P_3$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta. \tag{2.6}$$

where Δ is the area of the reference triangle ABC.

Given points $P = (p, q, r)$ and $U = (u, v, w)$. The Cevian quotient of P and U is the point

$$(u(-qru + rpv + pqw), v(-rpv + pqw + qru), w(-pqw + qru + rpv)). \tag{2.7}$$

Given a point $P(u, v, w)$, the Miquel associate of P is the point $M = (uM, vM, wM)$,

$$uM = \frac{a^2}{v + w}\left(-\frac{a^2vw}{v + w} + \frac{b^2wu}{w + u} + \frac{c^2uv}{u + v}\right),$$

$$(2.8)\qquad vM = \frac{b^2}{w+u}\left(-\frac{b^2wu}{w+u}+\frac{c^2uv}{u+v}+\frac{a^2vw}{v+w}\right),$$
$$wM = \frac{c^2}{u+v}\left(-\frac{c^2uv}{u+v}+\frac{a^2vw}{v+w}+\frac{b^2wu}{w+u}\right).$$

Given normalized points $P = (p,q,r)$ and $U = (u,v,w)$. The midpoint of P and U is the point $(\frac{p+u}{2}, \frac{q+v}{2}, \frac{r+w}{2})$, and the reflection of P in U is the point $(2u-p, 2v-q, 2w-r)$.

Given a point $P(u,v,w)$, the complement of P is the point $(v+w, w+u, u+v)$, the anticomplement of P is the point $(-u+v+w, -v+w+u, -w+u+v)$, the isotomic conjugate of P is the point (vw, wu, uv), and the isogonal conjugate of P is the point (a^2vw, b^2wu, c^2uv).

3. Haimov Triangle of an Arbitrary Point P

Theorem 3.1. *Given a point P with barycentric coordinates P=(u,v,w). Then the barycentric coordinates of the Haimov Triangle QaQbQc of P are as follows:*

$$uQa = -uwa^2 + uwc^2 - uva^2 + uvb^2 + c^2u^2 + u^2b^2 - a^2u^2 - a^2vw,$$
$$vQa = b^2(u+v)(u+v+w),$$
$$wQa = c^2(u+w)(u+v+w),$$
$$uQb = a^2(u+v)(u+v+w),$$
$$vQb = -uvb^2 + uva^2 + wc^2v - wb^2v - b^2v^2 + c^2v^2 - ub^2w + a^2v^2,$$
$$wQb = c^2(v+w)(u+v+w),$$
$$uQc = a^2(u+w)(u+v+w),$$
$$vQc = b^2(v+w)(u+v+w),$$
$$wQc = -uwc^2 + uwa^2 - wc^2v + wb^2v - w^2c^2 + w^2b^2 - vuc^2 + a^2w^2,$$

where Qa = (uQa,vQa,wQa), Qb = (uQb,vQb,wQb) and Qc = (uQc,vQc,wQc).

Theorem 3.2 below is given by Haim Haimov [9], Theorems 1 and 2. Haimov gives a synthetic proof of Theorem 3.2. Below we give a new proof which uses barycentric coordinates.

Theorem 3.2. *The Haimov Triangle and Triangle ABC are perspective and the Perspector is the Isogonal Conjugate of the Complement of Point P.*

We call the *Haimov Point of Point P* the Isogonal Conjugate of the Complement of Point P.

Figure 2 illustrates Theorem 3.2. In figure 2, P is a point of Ceva and $QaQbQc$ is the Haimov triangle of P. Then lines AQa, BQb and CQc concur in a point. The point of intersection of these lines is the Haimov point of P, the Isogonal Conjugate of the Complement of Point P.

For the definition of the Miquel associate of a point, see e.g. [19].

Theorem 3.3. *The Haimov Point of Point P is the Ceva Quotient of Point P and the Miquel associate of Point P.*

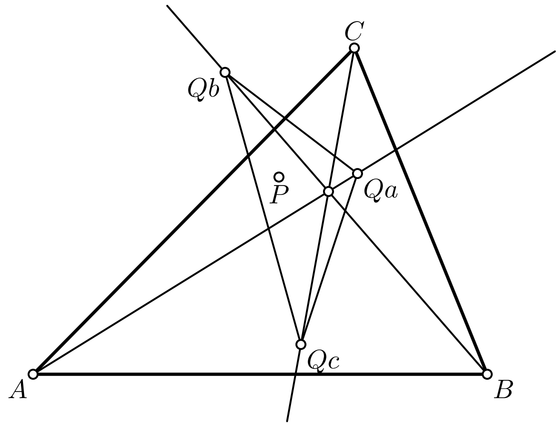

FIGURE 2.

Theorem 3.4. *The Haimov Triangle and the Cevian Triangle of Point P are perspective. The perspector K = (uK,vK,wK) has barycentric coordinates:*

$$uK = (uwa^2 - uwc^2 + uva^2 - uvb^2 - c^2u^2 - u^2b^2 + a^2u^2 + a^2vw)$$
$$(-b^2w^2u - b^2wvu - vc^2uw - v^2c^2u + 2\,a^2wvu + a^2w^2v + a^2wv^2),$$
$$vK = (-uvb^2 + uva^2 + wc^2v - b^2wv - b^2v^2 + v^2c^2 - ub^2w + a^2v^2)$$
$$(a^2w^2v - b^2w^2u - b^2wu^2 + a^2wvu - 2\,b^2wvu + vc^2uw + vc^2u^2),$$
$$wK = (-uwc^2 + uwa^2 - wc^2v + b^2wv - w^2c^2 + b^2w^2 - vc^2u + a^2w^2)$$
$$(a^2wv^2 - v^2c^2u - vc^2u^2 + a^2wvu + b^2wvu - 2\,vc^2uw + b^2wu^2).$$

4. PROOFS OF THE THEOREMS

Proof of theorem 3.1. Given $\triangle ABC$ with side lengths $a = BC, b = CA$ and $c = AB$. Let $P = (u, v, w)$ and let $A_1B_1C_1$ be the Cevian triangle of P.

The midpoint M_{AB} of segment AB has barycentric coordinates $(1, 1, 0)$. The line L_{AB} through points A and B has equation $z = 0$. Then the line L_1 through point M_{AB} and perpendicular to line L_{AB} has equation

$$c^2x - c^2y + (a^2 - b^2)z = 0.$$

The midpoint M_{AB_1} of segment AB_1 is $(2u + w, 0, w)$ and the equation of the line AB_1 is $y = 0$. Then the equation of the line L_2 through midpoint M_{AB_1} and perpendicular to the line AB_1 is as follows:

$$b^2wx + (-w^2c^2 + w^2a^2 + uwa^2 - uwb^2 - uwc^2)y - (2b^2uw + b^2w)z = 0.$$

Now the center O_1 of circle (ABB_1) is the intersection of the lines L_1 and L_2. Analogously, we find the center O_2 of circle (ACC_1). The reflection of point A in the line through points O_1 and O_2 is the vertex Qa of $\triangle ABC$. The barycentric coordinates of Qa are given in the statement of the theorem.

Analogously, we find the barycentric coordinates of the vertices Qb and Qc of triangle $\triangle QaQbQc$. $\square$.

Proof of Theorem 3.2. Let $P = (u, v, w)$ be a point and $Q = QaQbQc$ be the Haimov triangle of P. The line L_1 through points A and Qa has equation

$$L_1:\ c^2(u + w)y - b^2(u + v)z = 0.$$

The line line L_2 through points B and Qb has equation

$$L_2 : \; c^2(v+w)x - a^2(u+v)z = 0,$$

and the line L_3 through points C and Qc has equation

$$L_3 : \; b^2(v+w)y - a^2(u+w)z = 0.$$

The lines L_1, L_2 and L_3 concur in a point. Hence, triangles ABC and $QaQbQc$ are perspective.

The perspector Q of triangles ABC and $QaQbQc$ is the intersection of lines L_1 and L_2:

$$Q = \left(\frac{a^2}{v+w}, \frac{b^2}{w+u}, \frac{c^2}{u+v} \right)$$

Now, it is easy we to see that Q is the Isogonal Conjugate of the Complement of Point P. □

Proof of Theorem 3.3. Point Q in the proof of the previous theorem coincides with the Ceva Quotient of Point P and the Miquel associate of Point P. □

Proof of Theorem 3.4. Let $P = (u, v, w)$ be a point and let $A_1B_1C_1$ be the Cevian triangle of P. Let $Q = QaQbQc$ be the Haimov triangle of P. We find the line L_1 through points A_1 and Qa, the line L_2 through points B_1 and Qb, and the line L_3 through points C_1 and Qc.

The lines L_1, L_2 and L_3 concur in a point. Hence, triangles $A_1B_1C_1$ and $QaQbQc$ are perspective.

The perspector K of triangles $A_1B_1C_1$ and $QaQbQc$ is the intersection of lines L_1 and L_2. The barycentric coordinates of K are given in the statement of the theorem. □

5. Haimov Points

Haim Haimov [9] has identified as remarkable points of the triangle the Haimov points of the Incenter, Centroid and Circumcenter ([9], Theorem 3-5). He has suggested in [9] the "Discoverer" to investigate other Haimov Points and to identify them as remarkable points of the triangle. This section is answer to the Haimov's suggestion.

We have selected 200 remarkable points form the database of the "Discoverer". The result is as follows:

The "Discoverer" has identified 46 of these points as points available in Kimberling's ETC [10]. The rest of 154 points are remarkable points which are not available in [10]. The results are enclosed as supplementary material to this paper. See the enclosed folder "Haimov Points". The enclosed lists and tables are created by the "Discoverer"'. Every item in these lists or tables rewrites to theorem. For example, the first row of Table P-X rewrites to the following theorem:

Theorem 5.1. *The Haimov Point of the Incenter is the Isogonal Conjugate of the Spieker Center, which is point X(58) in* [10].

We could select from the database of the "Discoverer" more points, e.g. 1000 or 10000 points, so that we could extend the above result.

The 154 Haimov Points which are not available in [10] are probably new remarkable points. By using Theorem 3.2, we could easily find the barycentric coordinates of these points. The "Discoverer" could investigate the properties of these new remarkable points. Below is an example. Th Feuerbach Perspector is point X(12) in [10]. About the Ceva product see e.g. [4], Definitions, Constructions, Ceva Product.

Theorem 5.2. *The Haimov Point of the Feuerbach Perspector is the Ceva Product of the Apollonius Point and the Symmedian Point.*

We recommend the reader to prove the above theorem.

6. SPECIAL CASE: P = CENTROID

6.1. Barycentric Coordinates of the Haimov Triangle of the Centroid.

Below we give selected theorems about the Haimov triangle of the Centroid, discovered by the "Discoverer".

Theorem 6.1. *The barycentric coordinates of the Haimov Triangle QaQbQc of the Centroid are as follows:*

$$Qa = (b^2 + c^2 - 2a^2, 3b^2, 3c^2),$$
$$Qb = (3a^2, a^2 + c^2 - 2b^2, 3c^2),$$
$$Qc = (3a^2, 3b^2, a^2 + b^2 - 2c^2).$$

The above barycentric coordinates are given without proof by Haimov [9]. The above theorem is a corollary to Theorem 3.1.

6.2. Triangles perspective with the Haimov Triangle of the Centroid.

Haim Haimov [9] has suggested the "Discoverer" to find triangles which are perspective with the Haimov triangle of the Centroid. Below we answer the Haimov's suggestion.

Theorem 6.2. *The Haimov Triangle of the Centroid and Triangle ABC are perspective. The Perspector is the Symmedian Point.*

The above theorem is given by Haimov [9], Theorems 1 and 5. The proofs of Haimov are synthetic. Note that Theorem 6.2 is a corollary to our Theorem 3.2.

Below we give 15 additional triangles which are perspective with the Haimov triangle of the Centroid. These additional triangles are discovered by the "Discoverer".

Theorem 6.3. *The Haimov Triangle of the Centroid and the Medial Triangle are perspective. The perspector is the point X(2482),* [10].

Theorem 6.4. *The Haimov Triangle of the Centroid and the Orthic Triangle are perspective. The perspector is the point X(468),* [10].

Theorem 6.5. *The Haimov Triangle of the Centroid and the Symmedial Triangle are perspective. The perspector is the Symmedian Point.*

Theorem 6.6. *The Haimov Triangle of the Centroid and the Tangential Triangle are perspective. The perspector is the Symmedian Point.*

Theorem 6.7. *The Haimov Triangle of the Centroid and the Inner Grebe Triangle are perspective. The perspector is the Symmedian Point.*

Theorem 6.8. *The Haimov Triangle of the Centroid and the Outer Grebe Triangle are perspective. The perspector is the Symmedian Point.*

Theorem 6.9. *The Haimov Triangle of the Centroid and the Second Brocard Triangle are perspective. The perspector is the Symmedian Point.*

Theorem 6.10. *The Haimov Triangle of the Centroid and the Anticevian Triangle of the Reflection of the Centroid in the Symmedian Point are perspective. The perspector is the Centroid.*

Theorem 6.11. *Perspector of the Haimov Triangle of the Centroid and the Triangle of Reflections of the Symmedian Point in the Sidelines of Triangle ABC are perspective. The perspector is the Centroid.*

Theorem 6.12. *Perspector of the Haimov Triangle of the Centroid and the Stevanovic Triangle of the Symmedian Points of the Triangulation Triangles of the Centroid are perspective. The perspector is the Centroid.*

Theorem 6.13. *Perspector of the Haimov Triangle of the Centroid and the Euler Triangle of the Symmedian Point are perspective. The perspector is the Symmedian Point.*

Theorem 6.14. *The Haimov Triangle of the Centroid and the Honsberger Triangle are perspective. The perspector is not in the Kimberling's ETC* [10].

Theorem 6.15. *The Haimov Triangle of the Centroid and the Circum-Anticevian Triangle of the Parry Point are perspective. The perspector is not in the Kimberling's ETC* [10].

Theorem 6.16. *The Haimov Triangle of the Centroid and the Monge Triangle of the Miquel Antipedal Circles of the Centroid are perspective. The perspector is not in the Kimberling's ETC* [10].

Theorem 6.17. *The Haimov Triangle of the Centroid and the External Moses Triangle of the Excircles and the Circumcircle are perspective. The perspector is not in the Kimberling's ETC* [10].

We recommend the reader to prove the above theorem 6.3-6.17.

6.3. **Constructions of the Haimov Triangle of the Centroid.** We will consider the construction of the Haimov triangle of the Centroid by using compass and ruler. We will use the method described in [5].

In the construction below we will use theorems 6.2 and 6.11.

Let $QaQbQc$ be the Haimov Triangle of the Centroid. Denote by G the Centroid and by K the Symmedian Point of $\triangle ABC$. Denote by Ra the reflection of point K at line BC, and define analogously points Rb and Rc. Then the lines KA and GRa concur in point Qa, the lines KB and GRb concur in point Qb and the lines KC and GRc concur in point Qc.

Figure 3 illustrates the construction of the Haimov triangle of the Centroid. In figure 3, G is the Centroid, K is the Symmedian Point, Ra is the reflection of point K at line BC, Rb is the reflection of point K at line CA, Rc is the reflection of point K at line AB, and $QaQbQc$ is the Haimov triangle of the Centroid. Then the lines GRa and KA concur in point Qa, the lines GRb and KB concur in point Qb and the lines GRc and KC concur in point Qc.

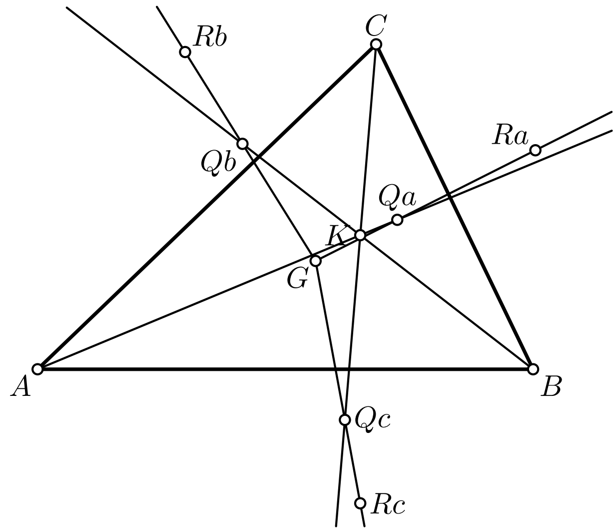

FIGURE 3.

In general, if we want to construct triangle $T = TaTbTc$, we may proceed as follows (see [5]): Let T and $XaXbXc$ be triangles which are perspective with perspector X, and let T and $YaYbYc$ be triangles which are perspective with perspector Y. Then the line XXa and YYa concur in point Ta, the lines XXb and YYb concur in point Tb and the lines XXc and YYc concur in point Tc.

Above we have used two of the theorems 6.2-6.17. Every pair of these theorems gives a construction of the Haimov triangle of the Centroid.

Problem 6.1. *How may ways could we use in order to construct the Haimov triangle of the Centroid, if we use the above method and the results of the previous section?*

Problem 6.2. *Find the Lazarov-Tabov complexity (see* [11],[16],[6]*) of all constructions of the Haimov triangle of the Centroid.*

Problem 6.3. *By using the above construction of the Haimov triangle of the Centroid, find the barycentric coordinates of this triangle.*

7. CONCLUSION

The computer program "Discoverer" fills a gap in the existing set of educational tools. It provides the possibility the students easily to discover new theorems in Euclidean geometry. By using the principles of "Discoverer", a number of similar computer programs could be created for the areas of high school teaching: physics, chemistry, biology, and so on. These computer programs could be considered as tools for activation of the interest of the students. We call the use of "Discoverer" for educational purposes "learning through discovery". We may consider the "learning through discovery" as a new important direction within the "learning through inquiry".

SUPPLEMENTARY MATERIAL

The enclosed file `http://www.journal-1.eu/2015/01/zips/Haimov_Points.zip` contains the files quoted in this paper.

References

[1] P. Douillet, *Translation of the Kimberling's Glossary into barycentrics*, 2012, http://www.ddekov.eu/e2/htm/links/Douillet.pdf

[2] Francisco Javier García Capitán, Barycentric Coordinates, in this number of this journal.

[3] S. Grozdev and D. Dekov, *A Survey of Mathematics Discovered by Computers* International Journal of Computer Discovered Mathematics, vol. 0, 2015, no. 0, 3-20.

[4] S. Grozdev and D. Dekov, Computer-Generated Encyclopeda of Euclidean Geometry, http://www.ddekov.eu/e2/index.htm

[5] S. Grozdev and D. Dekov, *Computer-generated mathematics: A note on the Haimov triangle* (Bulgarian), Mathematics and Informatics, vol. 57, 2014, no.6, 7-15. Enclosed File: http://www.ddekov.eu/papers/2014-6%20Haimov.zip

[6] S. Grozdev and D. Dekov, *The Computer improves the Steiner's Construction of the Malfatti Circles*, Mathematics and Informatics, 2015, vol. 58, no.1, 40-51.

[7] S. Grozdev and V. Nenkov, *Three Remarkable Points on the Medians of a Triangle* (Bulgarian), Sofia, Archimedes, 2012.

[8] S. Grozdev and V. Nenkov, *On the Orthocenter in the Plane and in the Space* (Bulgarian), Sofia, Archimedes, 2012.

[9] H. Haimov, A generzlization of the orthic triangle (Bulgarian), Mathejmatics and Informatics, to appear.

[10] C. Kimberling, *Encyclopedia of Triangle Centers - ETC*, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[11] B. Lazarov and J. Tabov, *Evaluation of algorithms for geometric construction* (Bulgarian), Education in mathematics and informatics, 1988, no. 6, pp.1-4.

[12] Gerry Leversha, *The Geometry of the Triangle*, The United Kingdom Mathematical Trust, The Pathways Series no.2, 2013.

[13] G. Paskalev and I. Tchobanov, *Remarkable Points in the Triangle* (in Bulgarian), Sofa, Narodna Prosveta, 1985.

[14] G. Paskalev, *With coordinates in Geometry* (in Bulgarian), Sofa, Modul-96, 2000.

[15] M. Schindler and K.Cheny, Barycentric Coordinates in Olympiad Geometry, 2012, http://www.mit.edu/~evanchen/handouts/bary/bary-full.pdf

[16] J. Tabov and B. Lazarov, *Geometric Constructions*, Sofia, Narodna Prosveta, 1990.

[17] E. W. Weisstein, *MathWorld - A Wolfram Web Resource*, http://mathworld.wolfram.com/

[18] *Wikipedia*, https://en.wikipedia.org/wiki/

[19] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf